On a Class of Multiplicity-Free Nilpotent $K_{\mathbb{C}}$-Orbits

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1. Introduction

An action of an algebraic reductive group G on an affine variety M is called *multiplicity-free* if the multiplicity of any particular irreducible representation of G in the space $\mathbb{C}[M]$ of regular functions on M is at most one: In [1], Kac provides a complete list of multiplicity-free actions for the case when G is a connected reductive algebraic group and M is a finite-dimensional vector space upon which G acts by an irreducible representation. Kac studied this case primarily to get an accounting of the possibilities for the action of $[\mathfrak{g}_0, \mathfrak{g}_0]$ on $\mathfrak{g}_i$, where $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded semisimple Lie algebra over $\mathbb{C}$.

In [2], Kato and Ochiai employ Kac's classification to develop a formula for the algebraic-geometric degree of K-invariant, multiplicity-free, affine variety V where K is a connected reductive complex algebraic group, V is a finite-dimensional representation of K such that $\mathbb{C}[V]$ is multiplicity-free and such that the image of K in $GL(V)$ contains all nonzero scalar matrices (this last property is used to provide an intrinsic grading of $\mathbb{C}[V]$ as a K-module). Kato and Ochiai then procede to explicitly evaluate their formula for the case when V the holomorphic tangent space of a Hermitian symmetric space G/K. In this situation, there is a element Z in the center of $\mathfrak{k}$, and a set of linearly independent dominant weights $\{\varphi_1, \ldots, \varphi_n\}$ such that

$$\mathbb{C}[V] \cong \bigoplus_{m\in\mathbb{N}^n} V_{m_1\varphi(Z)+\cdots+m_n\varphi(Z)}$$

Moreover, in this situation, there is a natural way of constructing the weights φ_i from set of strongly orthogonal non-compact roots $\{\gamma_1, \ldots, \gamma_n\}$, as well as an explicit accounting of the roots that contribute, via the Weyl dimension formula, to the degree of the orbit: it happens that the restrictions of the roots of Δ_M^+ break up into two disjoint subsets

$$\Delta^+_{short} = \left\{ \frac{1}{2}\gamma_i \mid 1 \le i \le n \right\} \text{ with common multiplity } r \text{ depending only on the choice of } G$$

$$\Delta^+_{long} = \left\{ \frac{1}{2}\gamma_i - \frac{1}{2}\gamma_j \mid 1 \le i < j \le n \right\} \text{ with common multiplicity } k \text{ depending only on the choice of } G$$

These facts allow Kato and Ochiai to reduce the problem of determining the algebraic-geometric degree of $\mathbb{C}[V]$ to the evaluation of a Selberg type integral

$$\int_{\mathcal{S}_n} \left(\prod_{i=1}^{r} x_i \right)^r \left(\prod_{i<j} (x_i - x_j) \right)^k d^n x$$

which in turn can be explicitly evaluated using a formula of Macdonald.

However, there is another interesting and important case of multiplicity-free actions: the case when M is an irreducible component of the associated variety of a multiplicity-free $(\mathfrak{g}, K)$-module. In this seminar, we shall reverse-engineer the construction of Kato and Ochiai to obtain a general class of multiplicity-free $K_{\mathbb{C}}$ orbits that we hope to eventually associate to corresponding familes of unipotent representations.[1] However, instead of starting with affine varieties known (from Kac's classification) to be multiplicity free,

[1] I'm thinking of "unipotent" as along the lines of the following conjecture of Vogan ([8]).

Conjecture 1.1. *Suppose $(\mathfrak{g},, K)$ is a reductive symmetric pair of Harish-Chandra class and $\mathcal{O} \subset \mathfrak{g}^*$ is a nilpotent coadjoint orbit. Assume that $\partial\mathcal{O}$ has codimension at least 4 in $\overline{\mathcal{O}}$. Suppose X is an irreducible $(\mathfrak{g}, K)$-module attached to $\mathcal{O}$. Then there is*

(i) *an element $\lambda \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$, and*
(ii) *an admissible representation χ of the stabilizer of $K(\lambda)$ of λ in K such that, as a representation of K*

$$X \cong Ind_{K(\lambda)}^{K}(\chi) \quad .$$

and happening to have an associated sequence of strongly orthogonal noncompact roots; we procede as follows:

- First of all, we start in the context of an arbitrary connected noncompact real semisimple Lie group G and introduce a means of constructing sequences $\{\gamma_1, \dots, \gamma_n\}$ of strongly orthogonal noncompact roots
- We then construct from these sequences corresponding families $\{\mathcal{O}_1, \dots, \mathcal{O}_n\}$ of $K_{\mathbb{C}}$-orbits such that
$$0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_n$$
- We show that the closure $\overline{\mathcal{O}_i}$ of each is multiplicity-free. and we explicitly identify the K-types of $\mathbb{C}\left[\overline{\mathcal{O}}\right]$ as
$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] \approx \bigoplus V_{a_1\gamma_1+\cdots+a_i\gamma_i}$$
where the sum is over the $a_i \in \mathbb{N}^i$ such that $a_1 \geq a_2 \geq \cdots \geq a_i \geq 0$.

Given these results, it then easy to

- observe that the degree of homogenity of a polynomial in $V_{a_1\gamma_1+\cdots+a_i\gamma_i}$ is $\sum_{j=1}^{i} a_j$ and so the filtration of $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ obtained by setting
$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right]_\ell \approx \bigoplus_{\substack{a_1\geq a_2\geq\cdots\geq a_i\geq 0\\ \sum a_j\leq \ell}} V_{a_1\gamma_1+\cdots+a_i\gamma_i}$$
provides a good filtration when regarded $\overline{\mathcal{O}_i}$ is interpreted as (a component of) the associated variety of a $(\mathfrak{g}, K)$-module.
- obtain a nice universal formulas for the degrees of the orbits in terms of generalized Selberg integrals of the sort I talked about last time.

In this way we recover Kato and Ochiai's results for the Hermitian symmetric case as well as our earlier results for the associated varieties of the unipotent representations associated with simple non-Euclidean Jordan algebras. We remark that in general these of multiplicity-free $K_{\mathbb{C}}$-orbits can not be predicted from Kac's classification of multiplicity-free actions because in general $\mathbb{C}[\mathfrak{p}]$ is not multiplicity-free.

2. Sequences of Strongly Orthogonal Noncompact Roots

Let G be a connected noncompact real semisimple Lie group. Let K be a maximal compact subgroup, θ a corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the corresponding Cartan decomposition of the complexification of the Lie algebra of G. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and extend it to a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Choose a positive system $\Delta^+(\mathfrak{t};\mathfrak{k})$ for $\Delta(\mathfrak{t};\mathfrak{k})$ and extend it to a positive system $\Delta^+(\mathfrak{h};\mathfrak{g})$ of $\Delta(\mathfrak{h};\mathfrak{g})$ in such a way that the highest weight $\beta \in \mathfrak{t}^*$ of the representation of K on $\mathfrak{p}$ coincides with the restriction to $\mathfrak{k}$ of the highest root of $\Delta^+(\mathfrak{h};\mathfrak{g})$ (which we also denote by β).[2] We construct maximal sequences $\{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal noncompact roots as follows.

- $\gamma_1 = \beta$
- γ_{i+1} is determined from $\{\gamma_1, \dots, \gamma_i\}$ by the requirements
  - (i) γ_{i+1} is in the orbit of β under the action of the Weyl group of K.
  - (ii) $\gamma_{i+1} \perp \gamma_j$ for $j = 1, , \dots, i$
  - (iii) $\omega_{i+1} = \sum_{j=1}^{i+1} \gamma_j \in \mathfrak{t}^*$ is dominant.

In the appendix, we tabulate the possibilities for simple noncompact Lie groups of classical type.

Remarks:

[2]In the Hermitian symmetric case where $\mathfrak{p}$ decomposes into a sum of two irreducibles, $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$, we take β to be the highest weight of the representation of K on, say, $\mathfrak{p}_+$.

- Occasionally, for a given $\mathfrak{g}$ we have several possibilities for sequences $\{\gamma_1, \dots, \gamma_n\}$. By imposing some additional requirements we could make the choice of sequence unique (e.g., demanding that the sequence terminate on the simple non-compact root).
- One could also consider relaxing the requirement that each γ_i lie in the K-Weyl orbit of the highest root; for example, one could simply require that each γ_i is a weight of the representation of K on $\mathfrak{p}$. This leads to more sequences of strongly orthogonal noncompact roots, but it seems that the sequences don't get any longer. This might be useful, nevertheless, for constructing some additional unipotent representations. (I'll talk about this at the end of the seminar.)
- In all but the case of $SU^*(2n)$ the maximal number of elements in a sequence of strongly orthogonal noncompact roots is equal to $\min(rank(G/K), rank(K))$. This suggests a connection with the number of commuting $\mathfrak{sl}(2,\mathbb{R})$ subalgebras of $\mathfrak{g}_0$.

  Indeed, to each $\gamma_i \in [\gamma_1, \dots, \gamma_n]$ we have an associated triple $\{x_i, h_i, y_i\}$ such that

  $$x_i \in \mathfrak{p}_{\gamma_i} \quad , \quad h_i \in i\mathfrak{t}_1 \subset i\mathfrak{k}_0 \quad , \quad y_i \in \mathfrak{p}_{-\gamma_i}$$

  In fact, one can arrange matters so that $y_i = \overline{x_i}$ and $\overline{h_i} = -h_i$. But then the Cayley transform

  $$\{x_i, h_i, y_i\} \to \{x_i', h_i', y_i'\} = \left\{ \frac{i}{2}(h_i - x_i + y_i), x_i + y_i, -\frac{i}{2}(h_i + x_i - y_i) \right\}$$

  will produce a subalgebra $\mathfrak{s}_i$ of $\mathfrak{g}_0$ that is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and moreover

  $$[\mathfrak{s}_i, \mathfrak{s}_j] = 0 \qquad , \qquad i \neq j$$

  Since the semisimple element h_i of the original triple $\{x_i, h_i, y_i\}$ lies in $i\mathfrak{k}_0$ and these all commute we must have $n \leq rank(K)$. On the other hand, since the semisimple element h_i' of the Cayley transform of $\{x_i, h_i, y_i\}$ is a semisimple element of $\mathfrak{p}$, we must have $n \leq rank(G/K)$. And so it's kind of interesting that in all cases except $SU^*(2n)$ we're getting the maximal possible (from this simple argument) number of commuting triples in $\mathfrak{g}_0$. It'd be interesting to see if this is also true for $SU^*(2n)$, where the number of γ_i is $\left[\frac{n}{2}\right]$, while $rank(K) = n-1$ and $rank(G/K) = n$. (In this case, strangely, the question seems to be why <u>don't</u> we have $\min(rank(G/K), rank(K))$ commuting $\mathfrak{sl}(2,\mathbb{R})$ subalgebras.)

3. Families of multiplicity-free $K_{\mathbb{C}}$-orbits

Let G be a noncompact real semisimple Lie group and let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a sequence of strongly orthogonal roots as constructed in the preceding section. We'll now associate to Γ a corresponding sequence $\{\mathcal{O}_1, \dots, \mathcal{O}_n\}$ of $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\mathfrak{p}}$.

We begin by choosing representative elements $x_i \in \mathfrak{p}_{\gamma_i} = \mathfrak{g}_{\gamma_i}$. As these are nilpotent elements of $\mathfrak{g}$, via a standard construction we can associate an S-triple; that is to say, we can find elements $h_i, y_i \in \mathfrak{g}$ so that for the triple $\{x_i, h_i, y_i\}$ the following commutation relations are satisfied:

$$[h_i, x_i] = 2x_i, \quad [h_i, y_i] = -2y_i \quad , \quad [x_i, y_i] = h_i \quad . \tag{1}$$

In fact, we can choose $y_i \in \mathfrak{p}_{-\gamma_i}$ and $h_i \in \mathfrak{t} \subset \mathfrak{k}$ so that $\{x_i, h_i, y_i\}$ is a normal triple in $\mathfrak{g}$; that is to say, $\{x_i, h_i, y_i\}$ satisfy both (1) and

$$\theta(x_i) = -x_i \quad , \quad \theta(y_i) = -y_i \quad , \quad \theta(h_i) = h_i \tag{2}$$

and $\mathfrak{s}_i = span_{\mathbb{R}}(x_i, h_i, y_i)$ is a θ-stable $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Moreover, since the γ_i's are strongly orthogonal, the corresponding $\mathfrak{s}_i$'s will be mutually centralizing; i.e., $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ if $i \neq j$.

We now set

$$\begin{aligned} X_i &= x_1 + x_2 + \cdots + x_i \\ H_i &= h_1 + h_2 + \cdots + h_i \\ Y_i &= y_1 + y_2 + \cdots + y_i \end{aligned}$$

and

$$\mathcal{O}_i = K_{\mathbb{C}} \cdot Y_i \subset \mathcal{N}_p.$$

$\{X_i, H_i, Y_i\}$ is easily seen to another normal S-triple in $\mathfrak{g}$.

We'll now show that the orbits $\mathcal{O}_i$, $i = 1, \ldots, n$ are all multiplicity-free.

Lemma 3.1. *For each $i = 1, \ldots, n$ the lowest eigenvalue of H_i in $\mathfrak{g}$ is -2. Moreover, if $z \in \mathfrak{g}$ and $[H_i, z] = -2z$, then $z \in span_{\mathbb{C}}(y_1, \ldots, y_i)$.*

Proof. Recall that in our construction the first element γ_1 in sequence of strongly orthogonal noncompact roots is always the highest weight of $\mathfrak{p}$ and that the subsequence elements $\gamma_2, \ldots, \gamma_n$ all lie in the (K-) Weyl orbit of γ_1. From Joseph [6] we know that decomposition of $\mathfrak{g}$ with respect to the eigenvalues of h_1 (i.e., the semisimple element of an $\mathfrak{sl}_2$ triple formed from a highest root vector) will look like

$$\begin{array}{lcccccccccccc}
x_1\bullet & & & & & & & & & & & & \\
 & * & * & * & * & * & * & & & & & & \\
h_1\bullet & & & & & & & \circ & \circ & \circ & \circ & \circ & \circ \\
 & * & * & * & * & * & * & & & & & & \\
y_1\bullet & & & & & & & & & & & &
\end{array}$$

That is to say, there'll be a single triplet corresponding to $\{x_1, h_1, y_1\}$, a bunch of pairs of doublets and a bunch of singlets. Since the roots γ_i are all in the Weyl group orbit of the highest root, for each i we can also choose a positive systems so that x_i is a highest root vector for $\mathfrak{g}$. It follows, just as above, that for each $i = 1, \ldots, n$, the smallest eigenvalue of h_i will be -2 and $[h_i, z] = -2z$ will imply that $z \in \mathbb{C}y_i$. Since the h_i are simultaneously diagonalizable we can conclude that the smallest eigenvalue of $H_i = h_1 + \cdots + h_i$ will be -2 and that if $[H_i, z] = -2z$ then we must have $z \in span_{\mathbb{C}}(y_1, \ldots, y_i)$. □

Because $H_i \in \mathfrak{t}$ is a semisimple element of $\mathfrak{k}$, it's stabilizer $\mathfrak{k}^{H_i}$ in $\mathfrak{k}$ is a reductive subalgebra. Let $\mathfrak{m}_i = \left[\mathfrak{k}^{H_i}, \mathfrak{k}^{H_i}\right]$ be the semisimple part of $\mathfrak{k}^{H_i}$.

Lemma 3.2. *$\mathfrak{m}_i \subset \mathfrak{k}^{Y_i}$.*

Proof. Let $\mathfrak{t}_1 = span_{\mathbb{C}}(h_1, \ldots, h_i)$ and let $\mathfrak{t}_0$ be the orthogonal complement of $\mathfrak{t}_1$ in $\mathfrak{t}$ (with respect to the Killing form). It is easy to see that $\mathfrak{m}_i$ is the subalgebra of $\mathfrak{k}$ generated by root spaces $\mathfrak{k}_\alpha$ such that $\alpha|_{\mathfrak{t}_1} = 0$. Suppose k_α is a root vector corresponding to a root space $\mathfrak{t}_\alpha \subset \mathfrak{m}_i$. We then have $[H_i, k_\alpha] = 0$ and so k_α will preserve the (-2)-eigenspace of H_i. By the preceding lemma

$$(-2)\text{-eigenspace of } H_i = span_{\mathbb{C}}(y_1, \ldots, y_i)$$

Because the y_i are also root vectors (for $\mathfrak{g}$) we must have

$$[k_\alpha, y_i] = c_{\alpha,i,j}\, y_j \qquad \text{for some } c_{\alpha,i,j} \in \mathbb{C} \quad .$$

But unless the $c_{\alpha.i,j} = 0$, γ_i would not be strongly orthogonal to γ_j; for otherwise we would have $\gamma_i - \gamma_j = \alpha \in \Delta(\mathfrak{h}; \mathfrak{g})$. We conclude that Y_i commutes with every root vector in $\mathfrak{m}_i$ and so, since $\mathfrak{m}_i$ is semisimple, $[Y_i, \mathfrak{m}_i] = 0$. □It now follows readily from results of Servedio [7] and Kimel'fel'd-Vinberg [5] that the $K_{\mathbb{C}}$-orbit through Y_i is multiplicity free.[3] However, by appealing to Kostant's statement of algebraic Frobenius reciprocity, we can not only prove that $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ is multiplicity-free, we can also identify the K-types.

Theorem 3.3 (Kostant, [3])**.** *Suppose G is a reductive algebraic group and V is an irreducible G-module. For $x \in V$, let G^x denote the stabilizer of x in G. If we denote by O_x the G-orbit through x, and by $\mathbb{C}(\mathcal{O}_x)$ the ring of everywhere-defined rational functions on $\mathcal{O}_x$, then if V_λ, $\lambda \in \widehat{G}$, is the representation space of an irreducible finite-dimensional representation of G*

$$\textit{multiplicity of } \lambda \textit{ in } \mathbb{C}(\mathcal{O}_x) \;=\; \dim \widetilde{V_\lambda}^{G^x}$$

where $\widetilde{V_\lambda}$ is the dual module of V_λ and $\widetilde{V_\lambda}^{G^x}$ is the space of vectors in $\widetilde{V_\lambda}$ that are fixed by G^x.

[3] A little more explicitly, the argument would procede as follows. It is easy to see that the tangent space to Y_i is generated by a parabolic subalgebra of $\mathfrak{k}$ corresponding to the non-negative eigenspaces of $ad_{\mathfrak{k}} H_i$. The preceding lemma then allows one to whittle this parabolic down to a Borel subalgebra. One then applies

Theorem ([5], [7]) Let K be a connected reductive algebraic group acting on an irreducible affine variety M. Then $\mathbb{C}[M]$ is multiplity-free if and only if there exists a Borel subgroup $B \subset K$ admitting an open orbit in M.

Remark: In the preceding theorem the stablilizer G^x of x need not be reductive.

Corollary 3.4. *$\mathbb{C}(\mathcal{O}_i)$ is multiplicity-free and V_λ is a K-type in $\mathbb{C}(\mathcal{O}_i)$ then $\lambda \in span_{\mathbb{R}}(\gamma_1, \ldots, \gamma_i)$.*

Proof. By Lemma 3.2, the stabilizer of Y_i in $\mathfrak{k}$ contains $\mathfrak{m}_i + \overline{\mathfrak{n}}_i$, where $\overline{\mathfrak{n}}_i$ is the direct sum of the negative eigenspaces of $ad(H_i)$ in $\mathfrak{k}$. Since all the negative roots of $\mathfrak{k}$ are contained in $\mathfrak{m}_i + \overline{\mathfrak{n}}_i$, a nonzero element of $\widetilde{V_\lambda}^{K^{Y_i}}$ will be a lowest weight vector that is also $\mathfrak{m}_i$-invariant. Algebraic Frobenious reciprocity then implies that if a K-type V_λ appears in $\mathbb{C}(\mathcal{O}_i)$ then the lowest weight vector of $\widetilde{V_\lambda}$ must be $\mathfrak{m}_i$-invariant. This in turn implies that $-\lambda$ (and so λ) is not supported on $\mathfrak{t}_0$. Thus, we must have

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \in \mathfrak{t}_1^*$$

And, of course, since the space of lowest weight vectors in $\widetilde{V_\lambda}$ will be 1-dimensional, algebraic Frobenius reciprocity also tells us that $\mathbb{C}(\mathcal{O}_i)$ is multiplicity-free. □

Proposition 3.5. *If V_λ is a K-types in $\mathbb{C}(\mathcal{O}_i)$, then its highest weight is of the form*

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$$

with $a_j \in \mathbb{Z}$ and

$$a_1 \geq a_2 \geq \cdots \geq a_i \geq 0 \ .$$

Proof. We first show that that coefficients a_k must be integers. Note that

$$\exp(i\pi h_j) \cdot Y_i = \exp(-2i\pi) Y_i = Y_i \qquad \text{for } 1 \leq j \leq i$$

and so $k_j \equiv \exp(i\pi h_j) \in K^{Y_i}$. On the other hand, $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ and $v_{-\lambda}$ is the lowest weight vector of $\widetilde{V_\lambda}$ we have

$$k_j \cdot v_{-\lambda} = \exp(-2i\pi a_j) v_{-\lambda} \ .$$

Thus, the lowest weight vector of $\widetilde{V_\lambda}$ will not be stablized by $k_i \in K^{Y_i}$ unless $a_j \in \mathbb{Z}$ for $j = 1, \ldots, i$.

It remains to prove the ordering of the coefficients a_i. Recall that for each i from 1 to n, $\omega_i = \gamma_1 + \cdots + \gamma_i$ is a dominant weight. It is clear that the ω_i are all linearly independent, and moreover,

$$\begin{aligned}
\gamma_1 &= \omega_1 \\
\gamma_2 &= \omega_2 - \omega_1 \\
&\vdots \\
\gamma_i &= \omega_i - \omega_{i-1}
\end{aligned}$$

And so

$$\begin{aligned}
\lambda &= a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \cdots + a_i\gamma_i \\
&= a_1\omega_1 + a_2(\omega_2 - \omega_1) + a_3(\omega_3 - \omega_2) + \cdots + a_i(\omega_i - \omega_{i-1}) \\
&= (a_1 - a_2)\omega_1 + (a_2 - a_3)\omega_2 + \cdots + (a_{i-1} - a_i)\omega_{i-1} + a_i\omega_i
\end{aligned}$$

Since the ω_i are all dominant and linearly independent, it is clear that λ can be dominant if and only if each $a_i \geq 0$ and $a_j - a_{j+1} \geq 0$ for $1 \leq j \leq i-1$. □Now actually we are most interested in the regular functions $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ supported on the closure of the orbit. Clearly,

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] \subset \mathbb{C}(\mathcal{O}_i)$$

We will now show that each of the K-types V_λ with λ satisfying the conditions of Lemma 3.5 actually occurs in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$.[4]

Theorem 3.6 (Kumar). *Let $\mathfrak{g}$ finite-dimensional semisimple Lie algebra, any pair λ, μ of dominant weights and any w in the Weyl group of $\mathfrak{g}$, the irreducible $\mathfrak{g}$-module $V_{\overline{\lambda+\omega\mu}}$ (for which $\lambda + w\mu$ is an extremal weight and $\overline{\lambda+\omega\mu}$ is the highest weight) occurs with multiplicity exactly one in the $\mathfrak{g}$-submodule $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ of $V_\lambda \otimes V_\mu$ (where e_λ and $e_{w\mu}$ are, respectively, weight vectors in the λ-weight space of V_λ and the $w\mu$-weight space of V_μ).*

[4]In other words, having proved the necessity of the conditions of Lemma 3.5, we will now prove their sufficiency.

Remark 3.7. The statement of the theorem is known as Kostant's strengthened Parthasarathy-Ranga Rao-Varadarajan conjecture.

Lemma 3.8. *Let $\omega_j = \gamma_1+\gamma_2+\cdots+\gamma_j$, $1 \le j \le i$. Then the K-type V_{ω_j} occurs in $S^j(\mathfrak{p})$ and the monomial $x_1 \cdots x_j \in S^j(\mathfrak{p})$ has a non-trivial projection onto the highest weight space of V_{ω_j}*

Proof. The case when $j = 1$ is trivial, since x_i is a highest weight vector of $\mathfrak{p}$. We now procede by induction on i. And all that requires is the preceding theorem with the identification of e_λ with the projection of $x_1 \cdots x_j$ onto the highest weight vector of V_{ω_j} (the inductive hypothesis) and the identification of $e_{w\mu}$ with x_{j+1} which by our construction always corresponds to the extremal weight of $\mathfrak{p}$. □

Theorem 3.9. *The K-type decomposition of $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ is exactly*

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] = \bigoplus_{\lambda\in\Lambda} V_\lambda$$

where

$$\Lambda = \{\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \mid a_i \in \mathbb{Z} \quad , \quad a_1 \ge a_2 \ge \cdots \ge a_i \ge 0\}$$

Proof. We have already seen that $\lambda \in \Lambda$ is a necessary condition for a K-type to be in $\mathbb{C}(\mathcal{O}_i)$ and so also to be in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$. We now note that the monomials $x_1 \cdots x_j$ are supported at Y_i; for

$$\begin{aligned}(x_1 \cdots x_j)(Y_i) &= \langle x_1, Y_i\rangle \cdots \langle x_j, Y_i\rangle = \langle x_1, y_1 + \cdots + y_i\rangle \cdots \langle x_j, y_1 + \cdots + y_i\rangle \\ &= \langle x_1, y_1\rangle \cdots \langle x_j, y_j\rangle \\ &= 1\end{aligned}$$

On the other hand, by the preceding lemma, a (suitably normalized) highest weight vector ϕ_{ω_j} of V_{ω_j} must be of the form

$$\phi_{\omega_j} = x_1 \cdots x_j \ + \ (\text{ other terms with at least one factor not among } x_1, \ldots, x_j)$$

The other terms will thus die upon evaluation at Y_i and so we must have

$$\phi_{\omega_j}(Y_i) = 1 \qquad \text{for} \quad 1 \le j \le i$$

But once we have the highest weight vectors of each V_{ω_j} supported at Y_i it is trivial to show that the products of these highest weight vectors will remain highest weight vectors and continue to be supported at $Y_i \in \mathcal{O}_i$. Thus, each of the K-types V_λ with

$$\lambda \in \Lambda' \equiv \{\alpha_1\omega_1 + \cdots + \alpha_i\omega_i \mid \alpha_i \in \mathbb{N}\}$$

will be supported at Y_i. The proof is completed by the observation that $\Lambda = \Lambda'$. □

4. Concluding Remarks

(1) In §2 we determined the restricted weights $\widetilde{\alpha} \in \Delta(\mathfrak{t}_1; \mathfrak{k})$ as well as their multiplicities. From this data and Kato-Ochiai's method, we can easily deduce integral formulas for algebraic-geometric degrees of the varieties $\overline{\mathcal{O}_i}$. The integral factor will look like

$$\int_{\mathcal{S}_j} \left(\prod_{j=1}^{i} x_j\right)^r \left(\prod_{1\le j<k\le i} (x_i - x_j)\right)^s d^i x$$

when the restricted root system as an A_i factor (i.e. in the Hermitian symmetric case) and

$$\int_{\mathcal{S}_j} \left(\prod_{j=1}^{i} x_j\right)^r \left(\prod_{1\le j<k\le i} (x_i^2 - x_j^2)\right)^s d^i x$$

otherwise.

(2) Here is a construction by which one might attach a certain unitary principal series representation to each of the orbits $\mathcal{O}_i$.
   - Take Cayley transform the triple $\{X_i, H_i, Y_i\}$ to get a triple $\left\{\widetilde{X_i}, \widetilde{H_i}, \widetilde{Y_i}\right\} \in \mathfrak{g}_0$.

- Use $\widetilde{H}_i$ form a real parabolic subgroup of G as well as a 1-parameter family of characters ν_s of the $A = \exp\left(\mathbb{R}\widetilde{H_i}\right)$.
- Form induced representations
$$I_s = Ind_{P=MAN}^{G}\left(1 \otimes \nu_s \otimes 1\right)$$
- Look for values of the parameter s where I_s degenerates. The expectation would be that we get a subrepresentation whose associated variety is one of the $\mathcal{O}_i$, and it may well be that many of these are unitary (a lot of bad K-types are going to drop out).

Appendix A

A.1. $G = SL(n, \mathbb{R})$, n even.

- $K = SO(n)$
- $rank\,(G/K) = n - 1$
- $rank\,(K) = [\frac{n}{2}]$
- $\beta = 2\omega_1$

There are two sequences of strongly noncompact orthogonal roots

- $\mathcal{S}_1 = \{[2, 0, \ldots, 0], [-2, 2, 0, \ldots, 0], \ldots, [0, \ldots, 0, -2, 2, 0, 0], [0, \ldots, 0, -2, 2, 2], [0, \ldots, 0, -2, 2]\}$
  $\Sigma = D_{[\frac{n}{2}]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \gamma_j\right\}$, multiplicity 1
- $\mathcal{S}_2 = \{[2, 0, \ldots, 0], [-2, 2, 0, \ldots, 0], \ldots, [0, \ldots, 0, -2, 2, 0, 0], [0, \ldots, 0, -2, 2, 2], [0, \ldots, 0, 2, -2]\}$
  $\Sigma = D_{[\frac{n}{2}]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \gamma_j\right\}$, multiplicity 1

A.2. $G = SL(2k+1, \mathbb{R})$, n odd.

- $K = SO(2k+1)$
- $rank\,(G/K) = 2k$
- $rank\,(K) = [\frac{n}{2}]$
- $\beta = 2\omega_1$

There is a single sequence of strongly noncompact orthogonal roots

- $\mathcal{S}_1 = \{[2, 0, \ldots, 0], [-2, 2, \ldots, 0], \ldots, [0, \ldots, -2, 2, 0], [0, \ldots, 0, -2, 4]\}$
  $\Sigma = B_{[\frac{n}{2}]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity 1
  $\Delta_{long} = \left\{\pm\frac{1}{2}\gamma_i \pm \gamma_j\right\}$, multiplicity 1

A.3. $G = SU^*(2n)$.

- $K = Sp(n, \mathbb{R})$
- $rank\,(G/K) = n - 1$
- $rank\,(K) = n$
- $\beta = \omega_2$

There are two sequences of strongly orthogonal noncompact roots

- $\mathcal{S}_1 = \begin{cases} \{[0,1,0,\dots,0],[0,-1,0,1,0,\dots,0],\dots,[0,\dots,-1,0,1]\} & \text{if } n \text{ is even} \\ \{[0,1,0,\dots,0],[0,-1,0,\dots,0],\dots,[0,\dots,0,-1,0,1,0]\} & \text{if } n \text{ is odd} \end{cases}$
  $\Sigma = C_{\left[\frac{n}{2}\right]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 4
  $\Delta_{long} = \{\pm\gamma_i\}$, multiplicity 3
- $\mathcal{S}_2 = \{[0,1,0,\dots,0],[2,-1,0,\dots,0]\}$
  $\Sigma = BC_2$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_1 \pm \frac{1}{2}\gamma_2\right\}$, multiplicity $2n-4$
  $\Delta_{middle} = \{\pm\gamma_i\}$
  $\Delta_{long} = \{\pm\gamma_1 \pm \gamma_2\}$, multiplicity 1

A.4. $G = SU(p,q)$, $p \leq q$**.**

- $K = S(U(p) \times U(q))$
- $rank(G/K) = p+q-1$
- $rank(K) = p$
- $\beta = \omega_1 + \omega_{p+q-1}$

There is a single sequence of strongly orthogonal noncompact roots

- $\mathcal{S} = \{[1,0,\dots,0,1],[-1,1,0,\dots,0,1,-1],\dots,[0,\dots,0,-1;0,\dots,1,-1,0,\dots,0]\}$
  (Here the last element is $-\omega_{p-1} + \omega_q - \omega_{q+1}$.)
  $\Sigma = BA_p$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity $p-q$
  $\Delta_{long} = \left\{\pm\frac{1}{2}(\gamma_i - \gamma_j)\right\}$, multiplicity 2

A.5. $G = SO(1,q)$ **,** q **even and** > 2**.**

- $K = SO(q)$
- $rank(G/K) = 1$
- $rank(K) = [\frac{n}{2}]$
- $\beta = \omega_1$

There are two possible sequences of $\left[\frac{q}{2}\right]$ strongly orthogonal noncompact roots

- $\mathcal{S}_1 = \{[1,0,\dots,0],[-1,1,0,\dots,0],\dots,[0,\dots,-1,1,1],[0,\dots,0,-1,1]\}$
  $\Sigma = D_{[q/2]}$
  $\Delta = \{\pm\gamma_i \pm \gamma_j\}$, multiplicity 1
- $\mathcal{S}_2 = \{[1,0,\dots,0],[-1,1,0,\dots,0],\dots,[0,\dots,-1,1,1],[0,\dots,0,1,-1]\}$
  $\Sigma = D_{[q/2]}$
  $\Delta = \{\pm\gamma_i \pm \gamma_j\}$, multiplicity 1

A.6. $G = SO(1,q)$ **,** q **odd and** > 2**.**

- $K = SO(q)$
- $rank(G/K) = 1$
- $rank(K) = [\frac{q}{2}]$
- $\beta = \omega_1$

There is a single sequence of strongly orthogonal noncompact roots

- $\mathcal{S} = \{[1,0,\ldots,0], [-1,1,0,\ldots,0], \ldots, [0,\ldots,0,-1,1,0], [0,\ldots,0,-1,2]\}$
  $\Sigma = B_n$
  $\Delta_{short} = \{\pm\gamma_i\}$, multiplicity 1
  $\Delta_{long} = \{\pm\gamma_i \pm \gamma_j\}$, multiplicity 1

A.7. $G = SO(2,q)$, $q > 2$**.**

- $K = S(O(p) \times O(q))$
- $rank(G/K) = 1$
- $rank(K) = [\frac{q}{2}]$
- $\beta = \omega_1$

There is a single sequence of 2 strongly orthogonal noncompact roots

- $\mathcal{S}_1 = [[1,0,\ldots,0], [-1,0,\ldots,0]]$
  $\Sigma = A_2$
  $\Delta = \{\pm(\gamma_1 - \gamma_2)\}$, multiplicity $(q-2)$

A.8. $G = SO(p,q)$ **,** $2 < p < q$ **,** p **even.**

- $K = S(O(p) \times O(q))$
- $rank(G/K) = p$
- $rank(K) = [\frac{p}{2}] + [\frac{q}{2}]$
- $\beta = \omega_1 + \omega_{[p/2]+1}$

There are four sequences of strongly orthogonal noncompact roots. To prescribe the first two, we introduce the following notation. Let $\sigma_{k,D,+}$, $\sigma_{k,D,-}$, and $\sigma_{k,B}$ be the sequences of $\left[\frac{k}{2}\right]$, $\left[\frac{k}{2}\right]$-dimensional vectors given by

$$\begin{aligned} \sigma_{k,D,+} &= \{[1,0,\ldots,0], [-1,1,0,\ldots,0], \ldots, [0,\ldots,0,-1,1,1], [0,\ldots,0,-1,1]\} \\ \sigma_{k,D.,-} &= \{[1,0,\ldots,0], [-1,1,0,\ldots,0], \ldots, [0,\ldots,-1,1,1], [0,\ldots,0,1,-1]\} \\ \sigma_{k,B} &= [1,0,\ldots,0], [-1,1,0,\ldots,0], \ldots, [0,\ldots,-1,1,0], [0,\ldots,0,-1,2] \end{aligned}$$

If $\ell \geq k$, we denote by $\{\sigma_{k,*} : \sigma_{\ell,*}\}$ the sequence of $\left[\frac{k}{2}\right]$, $\left(\left[\frac{k}{2}\right] + \left[\frac{\ell}{2}\right]\right)$-dimensional vectors formed by adjoining to each vector in $\sigma_{k,*}$ the corresponding element of $\sigma_{\ell,*}$.

- $\mathcal{S}_{1,+} = \begin{cases} \{\sigma_{p,D,+} : \sigma_{q,,D,*}\} & \text{if } q \text{ is even} \\ \{\sigma_{p,D,+} : \sigma_{q,B}\} & \text{if } q \text{ is odd} \end{cases}$
  $\Sigma = B_{[p/2]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity $(q-p)$
  $\Delta_{long} = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{1,-} = \begin{cases} \{\sigma_{p,D,-} : \sigma_{q,,D,*}\} & \text{if } q \text{ is even} \\ \{\sigma_{p,D,-} : \sigma_{q,B}\} & \text{if } q \text{ is odd} \end{cases}$
  $\Sigma = B_{[p/2]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity $(q-p)$
  $\Delta_{long} = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{2,+} = \{[1,0,\ldots,0;1,0,\ldots,0], [1,0\ldots,0;-1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_2\right\}$, multiplicity $q-2$

- $\mathcal{S}_{2,-} = \{[1,0,\ldots,0;1,0,\ldots,0], [-1,0\ldots,0;1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $q-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$

A.9. $G = SO(p,q)$ **,** $2 < p < q$ **,** p **odd.**

- $K = S(O(p) \times O(q))$
- $rank(G/K) = p$
- $rank(K) = [\frac{p}{2}] + [\frac{q}{2}]$
- $\beta = \omega_1 + \omega_{[p/2]+1}$

There are three sequences of strongly orthogonal noncompact roots.

- $\mathcal{S}_1 = \begin{cases} \{\sigma_{p,B} : \sigma_{q,,D,*}\} & \text{if } q \text{ is even} \\ \{\sigma_{p,B} : \sigma_{q,B}\} & \text{if } q \text{ is odd} \end{cases}$
  $\Sigma = B_{[p/2]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity $(q-p+2)$
  $\Delta_{long} = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{2,+} = \{[1,0,\ldots,0;1,0,\ldots,0], [1,0\ldots,0;-1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $q-2$
- $\mathcal{S}_{2,-} = \{[1,0,\ldots,0;1,0,\ldots,0], [-1,0\ldots,0;1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $q-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$

A.10. $G = SO(p,p)$ **,** $2 < p$**,** p **even.**

- $K = S(O(p) \times O(p))$
- $rank(G/K) = p$
- $rank(K) = p$
- $\beta = \omega_1 + \omega_{[p/2]}$

There are six sequences of strongly orthogonal noncompact roots.

- $\mathcal{S}_{+,+} = \left[\sigma_{1,[p/2],+}; \sigma_{[p/2],[q/2],+}\right]$
  $\Sigma = D_{[p/2]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{+,-} = \left[\sigma_{1,[p/2],+}; \sigma_{[p/2],[q/2],-}\right]$
  $\Sigma = D_{[p/2]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{-,+} = \left[\sigma_{1,[p/2],-}; \sigma_{[p/2],[q/2],+}\right]$
  $\Sigma = D_{[p/2]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{-,-} = \left[\sigma_{1,[p/2],-}; \sigma_{[p/2],[q/2],-}\right]$
  $\Sigma = D_{[p/2]}$
  $\Delta = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{2,+} = \{[1,0,\ldots,0;1,0,\ldots,0], [1,0\ldots,0;-1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$

$\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
$\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$

- $\mathcal{S}_{2,-} = \{[1,0,\ldots,0;1,0,\ldots,0], [-1,0\ldots,0;1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$

A.11. $G = SO(p,p)$ **,** $2 < p$ **,** p **odd.**

- $K = S(O(p) \times O(p))$
- $rank(G/K) = p$
- $rank(K) = p$
- $\beta = \omega_1 + \omega_{[p/2]}$

There are three sequences of strongly orthogonal noncompact roots:

- $\mathcal{S}_1 = \{[1,0,\ldots,0;1,0,\ldots0], [-1,1,0,\ldots,0;-1,1,0,\ldots,0], \ldots, [0,\ldots,0,-1,1;0,\ldots,0,-1,1,\ldots,0]\}$
  $\Sigma = B_{[p/2]}$
  $\Delta_{short} = \left\{\pm\frac{1}{2}\gamma_i\right\}$, multiplicity 2
  $\Delta_{long} = \left\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j\right\}$, multiplicity 2
- $\mathcal{S}_{2,+} = \{[1,0,\ldots,0;1,0,\ldots,0], [1,0\ldots,0;-1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
- $\mathcal{S}_{2,-} = \{[1,0,\ldots,0;1,0,\ldots,0], [-1,0\ldots,0;1,0,\ldots,0]\}$
  $\Sigma = D_2 = A_1 \times A_1$
  $\Delta_+ = \left\{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$
  $\Delta_- = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_2\right\}$, multiplicity $p-2$

A.12. $G = SO^*(2n)$ **,** n **even.**

- $K = U(n)$
- $rank(G/K) = [\frac{n}{2}]$
- $rank(K) = n$
- $\beta = \omega_2$

There is a single sequence of strongly orthogonal noncompact roots

- $\mathcal{S} = \begin{cases} \{[0,1,0,\ldots,0], [0,-1,0,1,0,\ldots,0], \ldots, [0,\ldots,0,-1,0,1]\} & \text{if } n \text{ is even} \\ \{[0,1,0,\ldots,0], [0,-1,0,1,0,\ldots,0], \ldots, [0,\ldots,0,-1,0,1,0]\} & \text{if } n \text{ is odd} \end{cases}$
  $\Sigma = A_n$
  $\Delta_{[\frac{n}{2}]} = \left\{\pm\left(\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_j\right)\right\}$, multiplicity 4

A.13. $G = SO^*(2n)$ **,** n **odd.**

- $K = U(n)$
- $rank(G/K) = [\frac{n}{2}]$
- $rank(K) = n$
- $\beta = \omega_2$

There is a single sequence of strongly orthogonal noncompact roots

- $\mathcal{S}=\begin{cases}\{[0,1,0,\ldots,0],[0,-1,0,1,0,\ldots,0],\ldots,[0,\ldots,0,-1,0,1]\} & \text{if } n \text{ is even}\\ \{[0,1,0,\ldots,0],[0,-1,0,1,0,\ldots,0],\ldots,[0,\ldots,0,-1,0,1,0]\} & \text{if } n \text{ is odd}\end{cases}$
  $\Sigma=AC_{\left[\frac{n}{2}\right]}$
  $\Delta_{short}=\left\{\frac{1}{2}\gamma_i\right\}$, multiplicity 2
  $\Delta_{long}=\left\{\frac{1}{2}\gamma_1\pm\frac{1}{2}\gamma_2\right\}$, multiplicity 4

A.14. $G=Sp\,(n)$.

- $K=U\,(n)$
- $rank\,(G/K)=n$
- $rank\,(K)=n$
- $\beta=2\omega_1$

There is a single sequence of strongly orthogonal noncompact roots

- $\mathcal{S}=\{[2,0,\ldots,0],[-2,2,0,\ldots,0],\ldots,[0,\ldots,0,-2,2],[0,\ldots,0,-2]\}$
  $\Sigma=A_n$
  $\Delta=\left\{\pm\left(\frac{1}{2}\gamma_i-\frac{1}{2}\gamma_j\right)\right\}$, multiplicity 1

A.15. $G=Sp\,(p,q)$, $(p\leq q)$.

- $K=Sp\,(p)\times Sp\,(q)$
- $rank\,(G/K)=p$
- $rank\,(K)=p+q-1$
- $\beta=\omega_1+\omega_{p+1}$

There are three sequences of strongly orthogonal noncompact roots.

- $\mathcal{S}_1=\{[1,0,\ldots,0;1,0,\ldots,0],[-1,1,0,\ldots,0;-1,1,0,\ldots,0],\ldots,[0,\ldots,-1,1;0,,\ldots,0,-1,1,0,\ldots,0]\}$
  $\Sigma=BC_p$
  $\Delta_{short}=\left\{\frac{1}{2}\gamma_i\right\}$, multiplicity $2\,(p-q)$, (if $p=q$ then $\Delta_{short}=\emptyset$)
  $\Delta_{middle}=\left\{\pm\frac{1}{2}\gamma_i\pm\frac{1}{2}\gamma_j\right\}$, multiplicity 2
  $\Delta_{long}=\{\gamma_i\}$ multiplicity 2
- $\mathcal{S}_{2,+}=\{[1,0,\ldots,0;1,0,\ldots,0],[1,0\ldots,0;-1,0,\ldots,0]\}$
  $\Sigma=D_2=A_1\times A_1$
  $\Delta_+=\left\{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2\right\}$, multiplicity $2\,(p-1)$
  $\Delta-=\left\{\frac{1}{2}\gamma_1-\frac{1}{2}\gamma_2\right\}$, multiplicity $2\,(q-1)$
- $\mathcal{S}_{2,-}=\{[1,0,\ldots,0;1,0,\ldots,0],[-1,0\ldots,0;1,0,\ldots,0]\}$
  $\Sigma=D_2=A_1\times A_1$
  $\Delta_+=\left\{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2\right\}$, multiplicity $2\,(q-1)$
  $\Delta_-=\left\{\frac{1}{2}\gamma_1-\frac{1}{2}\gamma_2\right\}$, multiplicity $2\,(p-1)$

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