Solution to Practice Problems / Midterm 2.

(1) We use the method of Lagrange multipliers,

$f(x,y,z) = xyz$, $g(x,y,z) = x + y + z^2 = 20$

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \\ g(x,y,z) = x+y+z^2 = 20 \end{cases} \qquad \begin{cases} yz = \lambda \cdot 1 \\ xz = \lambda \cdot 1 \\ xy = \lambda (2z) \\ x+y+z^2 = 20 \end{cases}$$

Since $x,y,z > 0$,

we can write $\frac{yz}{xz} = \frac{\lambda}{\lambda} = 1$, $x = y$.

$\frac{xz}{xy} = \frac{\lambda}{2\lambda z}$, $\frac{z}{y} = \frac{1}{2z}$, $2z^2 = y$, $z^2 = \frac{y}{2}$.

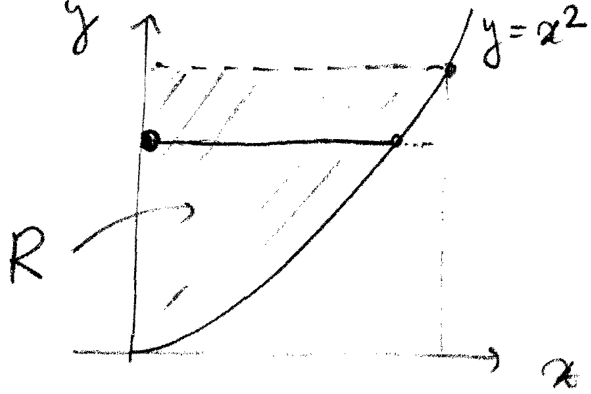
Hence $x+y+z^2 = 20$ becomes $y + y + \frac{y}{2} = 20$, $\boxed{y = 8}$.

$\boxed{x = y = 8}$, $z^2 = 4$, $z = \pm 2$, $\boxed{z = 2}$

The product is max for $\boxed{x = y = 8,\ z = 2}$

(2) $I = \int_0^1 \int_{x^2}^1 x e^{-y^2} dy\, dx$ cannot be computed in this order of integration since e^{-y^2} does not have an explicit anti-derivative. We try to invert the order of integration

$I = \iint_R x e^{-y^2} dA$, $R = \left\{ (x,y), \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le 1 \end{array} \right\}$,



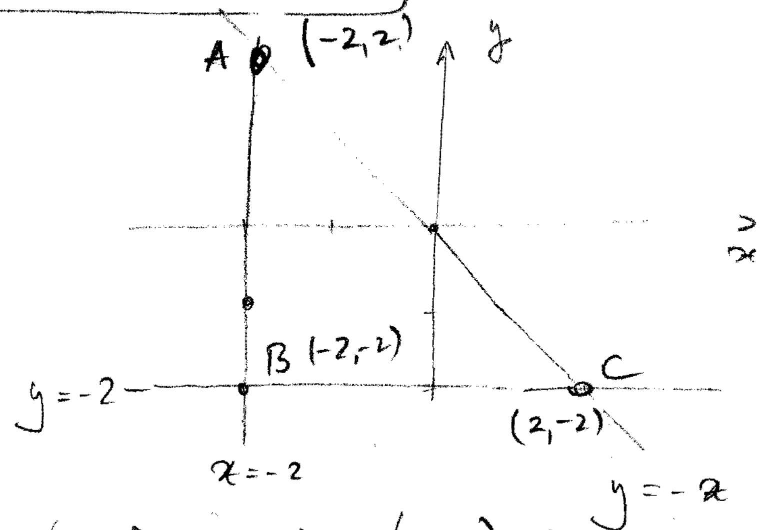
$$R=\left\{(x,y),\ \begin{array}{l}0\le x\le\sqrt{y}\\ 0\le y\le 1\end{array}\right\}$$ (2)

$$I=\int_0^1\int_0^{\sqrt{y}} x e^{-y^2}\,dx\,dy =$$

$$=\int_0^1 e^{-y^2}\left(\tfrac{1}{2}x^2\right)\Big|_{x=0}^{x=\sqrt{y}}dy=\int_0^1 \tfrac{1}{2}ye^{-y^2}dy=$$

$$=\left(-\tfrac{1}{4}e^{-y^2}\right)\Big|_{y=0}^{y=1}=\boxed{\left(-\tfrac{1}{4}\right)\left(e^{-1}-1\right)}$$

(3) $T(x,y)=x^2y-4y-4x+50.$



Critical points

$$\begin{cases}\dfrac{\partial T}{\partial x}=2xy-4=0\\[2mm] \dfrac{\partial T}{\partial y}=x^2-4=0\end{cases}$$

The solution are $(2,1)$ and $(-2,-1)$. But $(2,1)$ is outside of the domain, so the only critical point in the domain is $(-2,-1)$, $T(-2,-1)=58$.

We study the extrema of T on the boundary.

AB: $x=-2,\ -2\le y\le 2.$

$T(-2,y)=4y-4y+8+50=58.$ T is constant on AB.

AC: $y=-x,\ -2\le x\le 2.$

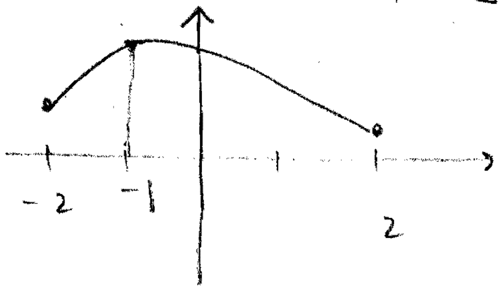
$T(x,-x)=-x^3+4x-4x+50=-x^3+50.$

Max point is $x = -2$ $(T = 58)$ and the min point is $x = 2$ $(T = 42)$.

$\underline{BC}$: $-2 \le x \le 2$, $\underline{y = -2}$, $T(x, -2) = -2x^2 + 8 - 4x + 50 =$

$= -2x^2 - 4x + 58$; Max point is $x = \underline{-1}$ $(T = 60)$

Min point is $x = 2$ $(T = 42)$



We compare all the values

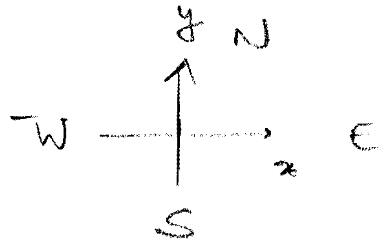
(x,y)	$T(x,y)$
$(-2,-1)$	58
$(-2,y)$	58
$(2,-2)$	42
$(-1,-2)$	60

The hottest point is $(-1,-2)$ at $T = 60$ and the coldest point is $(2,-2)$ at $T = 42$.

⑤ $z = f(x,y) = 100 - 2x^2 - y^2$, $\vec{\nabla} f = \langle -4x, -2y \rangle$;

$\vec{\nabla} f(2,3) = \langle -8, -6 \rangle$.

a) NW $\to$ $\vec{u} = \frac{1}{\sqrt{(-1)^2 + 1^2}} \langle -1, 1 \rangle =$



$= \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$D_{\vec{u}} f(2,3) = \vec{\nabla} f \cdot \vec{u} = \langle -8, -6 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{8-6}{\sqrt{2}} = \sqrt{2} > 0$

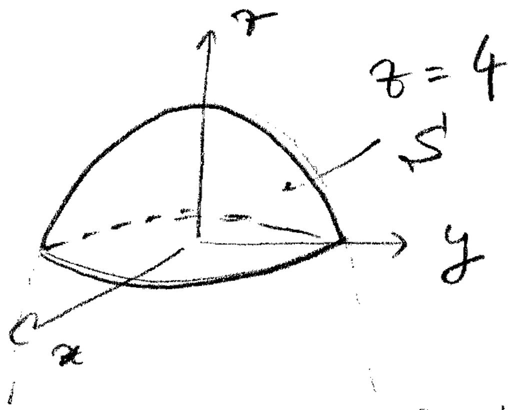
The ant is ascending, since the directional derivative of the height function is positive in that direction

b) We need to find $\vec{u}$ such that $D_{\vec{u}} f(2,3) = 0$, i.e. $\vec{\nabla} f(2,3)\cdot\vec{u}=0$, $\langle -8,-6\rangle\cdot\vec{u}=0$. (4)

One possible choice is $\vec{u}=\dfrac{1}{\sqrt{8^2+(-6)^2}}\langle 8,-6\rangle$,

$$\boxed{\vec{u}=\left\langle \frac{4}{5}, \frac{-3}{5}\right\rangle}$$

(4)



$z=4-x^2-y^2=f(x,y)$

$$A(S)=\iint_R \sqrt{1+(f_x)^2+(f_y)^2}\,dA$$

$$R=\{(x,y)\ /\ x^2+y^2\le 4\}$$

$$\begin{cases} f_x=-2x \\ f_y=-2y \end{cases}$$

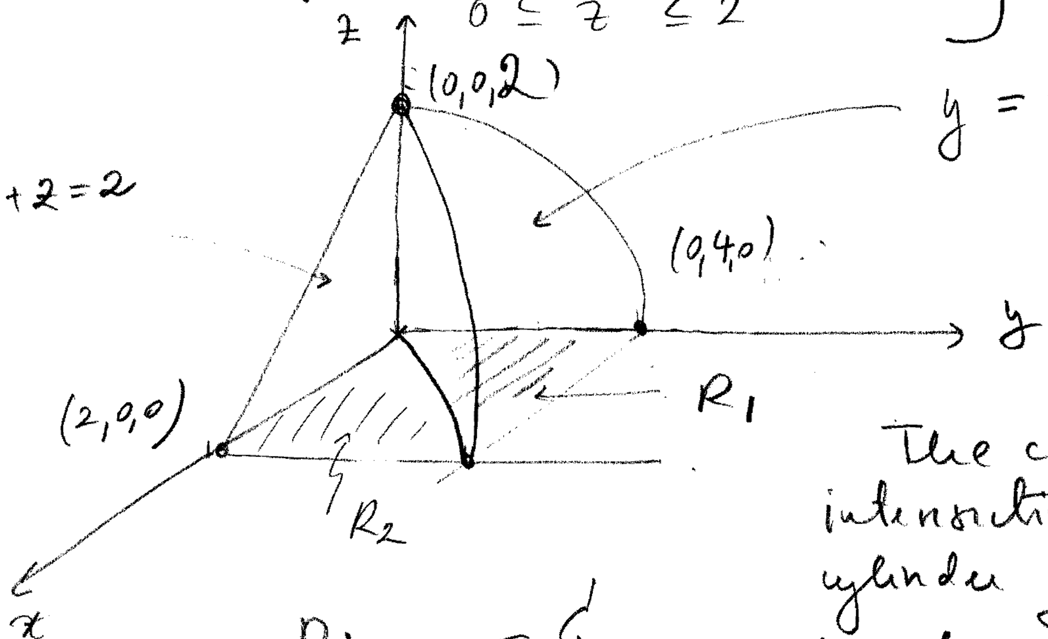
$$A(S)=\iint_R \sqrt{1+(-2x)^2+(-2y)^2}\,dA=\int_0^{2\pi}\int_0^{2}\sqrt{1+4r^2}\cdot r\,dr\,d\theta=$$

$$\underset{\substack{u=4r^2\\ du=8r\,dr}}{=} 2\pi\int_0^{16}\sqrt{1+u}\,\frac{du}{8}=\frac{\pi}{4}\cdot\frac{2}{3}\cdot(1+u)^{3/2}\Big|_{u=0}^{u=16}=$$

$$=\boxed{\frac{\pi}{6}\left(17^{3/2}-1\right)}$$

6) $I = \int_0^2 \int_0^{4-z^2} \int_0^{2-z} f(x,y,z)\, dx\, dy\, dz = \iiint_E f \, dV$

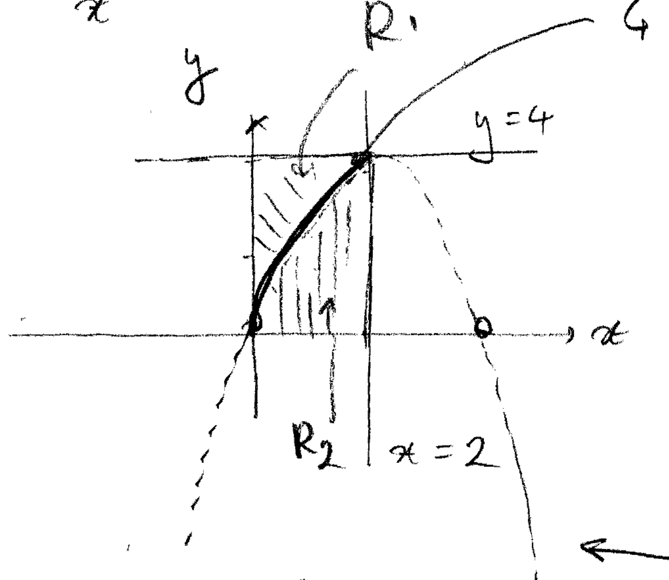
$E = \left\{ (x,y,z) \;/\; \begin{array}{l} 0 \le x \le 2-z \\ 0 \le y \le 4-z^2 \\ 0 \le z \le 2 \end{array} \right\}$



The curve of intersection of the cylinder $y = 4 - z^2$ and the plane $x+z=2$ satisfies $\begin{cases} y = 4 - z^2 \\ x + z = 2 \end{cases}$ $z = 2 - x$

The projection of this curve onto the xy-plane is the curve C given by

$y = 4 - (2-x)^2$ (a parabola!)



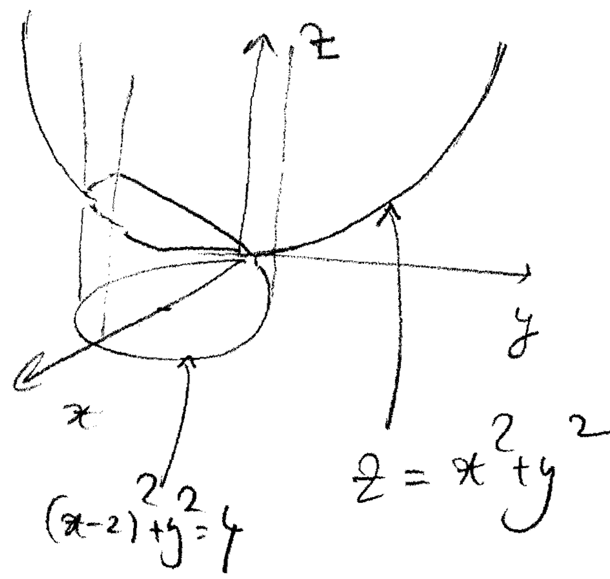
Hence:

$I = \iint_{R_1} \int_0^{\sqrt{4-y}} f \, dz\, dx\, dy + \iint_{R_2} \int_0^{2-x} f \, dz\, dx\, dy =$

$= \int_0^4 \int_0^{2-\sqrt{4-y}} \int_0^{\sqrt{4-y}} f \, dz\, dx\, dy + \int_0^4 \int_0^{2-\sqrt{4-y}} \int_0^{2-x} f \, dz\, dx\, dy$

(7) The cylinder $x^2+y^2=4x$ can be written

$$(x-2)^2+y^2=4$$



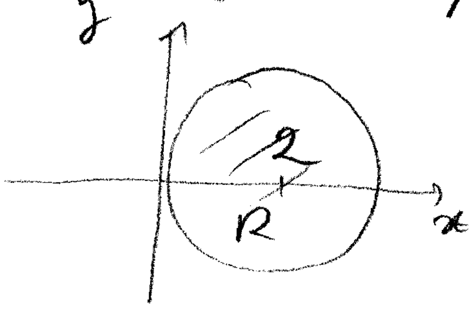
$$\text{Volume} = \iint_R (x^2+y^2)\, dA,$$

where $R=\{(x,y) \,/\, (x-2)^2+y^2 \le 4\}$

In polar coordinates the circle $(x-2)^2+y^2=4$, becomes

$$r^2 = 4r\cos\theta, \quad r = 4\cos\theta, \quad \text{so}$$

$$R=\left\{(r,\theta) \,/\, 0\le r \le 4\cos\theta,\ \frac{-\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$$



Hence:

$$\text{Volume} = \int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta} r^2 \cdot r\, dr\, d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_{r=0}^{r=4\cos\theta} = \int_{-\pi/2}^{\pi/2} \frac{1}{4}\, 4^4 \cos^4\theta\, d\theta =$$

$$= 64\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = 16\int_{-\pi/2}^{\pi/2}\left(1+2\cos(2\theta)+\cos^2(2\theta)\right)d\theta$$

$$= 16\left(\pi + \frac{1}{4}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos(4\theta)}{2}\right)d\theta =$$

$$= \boxed{24\pi}.$$

⑧ We find the critical points of f

$$\begin{cases} f_x = 4x^3 + 4y = 0 \\ f_y = 4y^3 + 4x = 0 \end{cases}, \quad \begin{cases} y = -x^3 \\ x = -y^3 \end{cases}$$

We can write that $x = -y^3 = -(-x^3)^3 = +x^9$

ie $x - x^9 = 0$, $x(x^8 - 1) = 0$ ie $\boxed{x=0, y=0}$ $\boxed{x=1, y=-1}$ $\boxed{x=-1, y=1}$

We have that

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = 4$

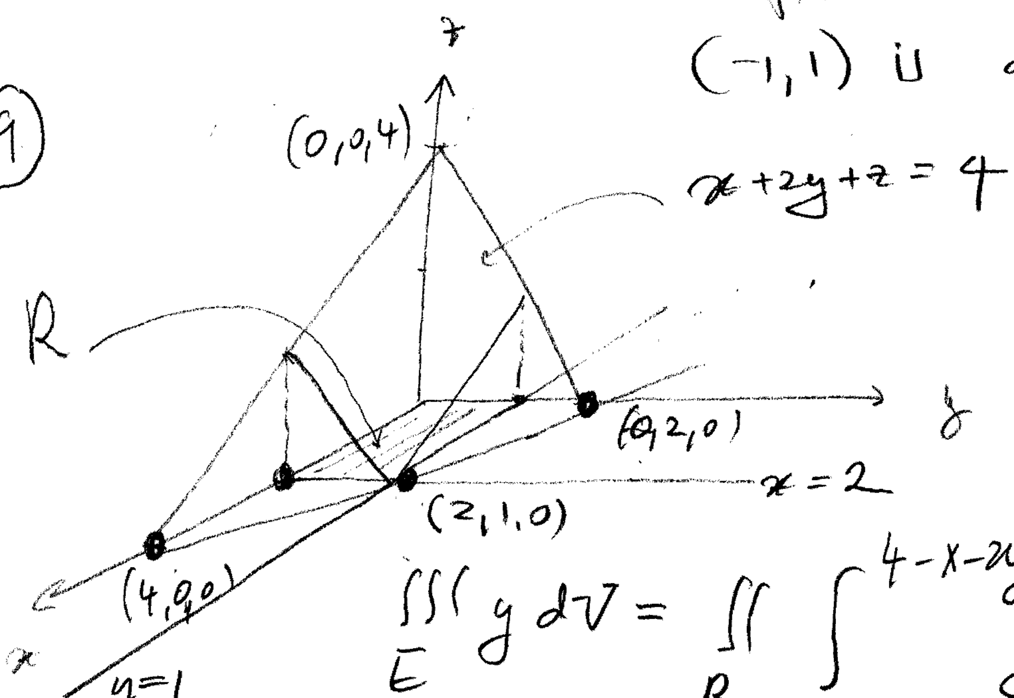
$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16.$

$D(0,0) = -16 < 0$, so $(0,0)$ is a saddle point

$D(1,-1) = 144 - 16 > 0$ and $f_{xx}(1,-1) = 12 > 0$, so $(1,-1)$ is a local min.

$D(-1,1) = 144 - 16 > 0$ and $f_{xx}(-1,1) = 12 > 0$, so $(-1,1)$ is a local min.

⑨



$$\iiint_E y \, dV = \iint_R \int_0^{4-x-2y} y \, dz \, dx \, dy,$$

$$= \int_0^1\int_0^2\int_0^{4-x-2y} y\,dz\,dx\,dy = \int_0^1\int_0^2 y(4-x-2y)\,dx\,dy$$ (8)

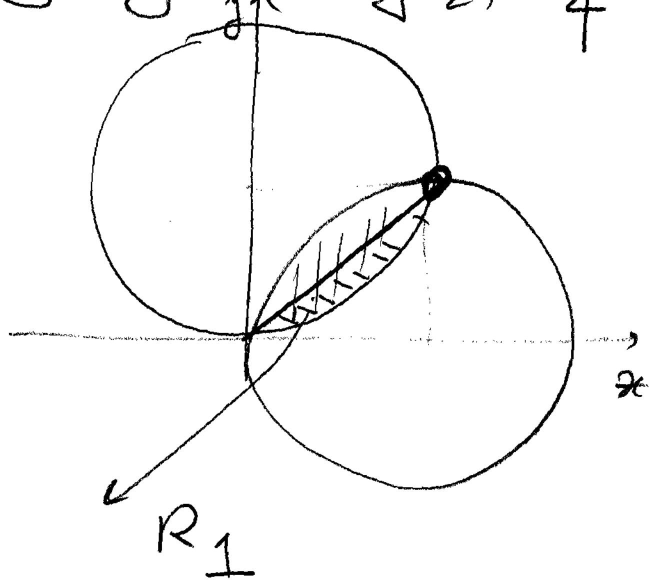
$$= \int_0^1\int_0^2 4y\,dx\,dy - \int_0^1\int_0^2 xy\,dx\,dy - 2\int_0^1\int_0^2 y^2\,dx\,dy =$$

$$= 2\cdot 2y^2\Big|_0^1 - \left(\tfrac12 x^2\Big|_0^2\right)\left(\tfrac12 y^2\right)\Big|_0^1 - 2\left(\tfrac13 y^3\right)\Big|_0^1\cdot 2 =$$

$$= 4 - 2\cdot\frac12 - \frac43 = \boxed{\frac53}$$

(10) $r=\cos\theta$, $r^2 = r\cos\theta$, $x^2+y^2=x$, $(x-\frac12)^2+y^2=\frac14$

$r=\sin\theta$, $r^2=r\sin\theta$, $x^2+y^2=y$, $x^2+(y-\frac12)^2=\frac14$



Area $= 2$ Area$(R_1) = 2\iint_{R_1} 1\,dA$

$R_1 = \left\{(r,\theta) \,/\, \begin{matrix} 0\le\theta\le\frac{\pi}{4} \\ 0\le r\le\sin\theta \end{matrix}\right\}$

$$\text{Area} = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta =$$

$$= \cancel{2}\int_0^{\pi/4}\frac{1}{\cancel{2}}\sin^2\theta\,d\theta = \int_0^{\pi/4}\frac{1-\cos(2\theta)}{2}\,d\theta =$$

$$= \frac{\theta}{2}\Big|_0^{\pi/4} - \frac14\sin(2\theta)\Big|_0^{\pi/4} = \frac{\pi}{8}-\frac14 = \boxed{\frac{\pi-2}{8}}$$