

Derivatives of Faber polynomials and Markov inequalities

Igor E. Pritsker*

*Department of Mathematics, 401 Mathematical Sciences, Oklahoma State
University, Stillwater, OK 74078-1058, U.S.A.
E-mail: igor@math.okstate.edu*

DEDICATED TO PROFESSOR D. GAIER ON THE OCCASION OF HIS 75TH BIRTHDAY

We study asymptotic behavior of the derivatives of Faber polynomials on a set with corners at the boundary. Our results have applications to the questions of sharpness of Markov inequalities for such sets. In particular, the found asymptotics are related to a general Markov-type inequality of Pommerenke and the associated conjecture of Erdős. We also prove a new bound for Faber polynomials on piecewise smooth domains.

Key Words: Faber polynomials, derivatives, asymptotics, Markov inequalities

1. FABER POLYNOMIALS AND THEIR DERIVATIVES

Let K be a compact connected set. Denote the unbounded connected component of $\overline{\mathbb{C}} \setminus K$ by Ω . Consider the canonical conformal mapping $\Psi : \Delta \rightarrow \Omega$, where $\Delta := \{w : |w| > 1\}$, with the Laurent expansion at ∞

$$\Psi(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \dots, \quad |w| > 1, \quad c > 0. \quad (1.1)$$

We note that $c = \text{cap}(K)$ is the logarithmic capacity of K . The Faber polynomials $\{F_n(z)\}_{n=0}^{\infty}$, $\deg F_n = n$, are defined via the Laurent expansion of the generating function (cf. [21] or [6])

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad z \in K, \quad |w| > 1. \quad (1.2)$$

*Research supported in part by the National Science Foundation grant DMS-9996410.

They proved to be of considerable importance in approximation theory (see, e.g., [6] and [20]), complex function theory [2] and orthogonal polynomials (cf. [22] and [20]).

An equivalent definition of Faber polynomials can be given by using the inverse conformal mapping $\Phi := \Psi^{-1}$. Then $F_n(z)$ is the polynomial part of the Laurent expansion of $\Phi^n(z)$ near $z = \infty$, i.e.,

$$\Phi^n(z) = F_n(z) + E_n(z), \quad z \in \Omega, \quad (1.3)$$

where

$$E_n(z) = O\left(\frac{1}{z}\right), \quad \text{as } z \rightarrow \infty.$$

If the boundary of Ω is sufficiently smooth, then it is possible to show that

$$\lim_{n \rightarrow \infty} E_n(z) = 0,$$

for $z \in \Omega$, and even for $z \in \partial\Omega$ (see [21, Ch. 4] and [20]). Thus we arrive at the classical asymptotics for Faber polynomials

$$F_n(z) = \Phi^n(z) + o(1), \quad n \rightarrow \infty, \quad (1.4)$$

where $z \in \bar{\Omega}$. Note that Faber polynomials typically tend to zero outside $\bar{\Omega}$, as $n \rightarrow \infty$ (cf. [21, Ch. 4] and [7]). Using standard methods, one can prove the following asymptotics for the derivatives of Faber polynomials.

PROPOSITION 1.1. *Suppose that $\partial\Omega$ is an analytic curve, so that Φ can be continued conformally through $\partial\Omega$. Then there exist a domain $\tilde{\Omega} \supset \bar{\Omega}$ and $r \in (0, 1)$ such that*

$$F_n^{(k)}(z) = \frac{d^k}{dz^k} (\Phi^n(z)) + O(r^n), \quad \text{as } n \rightarrow \infty, \quad (1.5)$$

for any $z \in \tilde{\Omega}$ and $k = 0, 1, 2, \dots$

These asymptotics may be viewed as the differentiated versions of equations (1.3) and (1.4). One can obtain a similar result, for the derivatives up to a certain order, in the case of sufficiently smooth (not analytic) boundary $\partial\Omega$. The ideas are close to those of [21, Ch. 4], but they require a much more technical argument than the proof of Proposition 1.1.

Asymptotics for Faber polynomials in the case of non-smooth boundary were obtained in [16]. If $\partial\Omega$ has the angle of opening $\alpha\pi$ at $z \in \partial\Omega$, $0 < \alpha \leq 2$, with respect to Ω , then (1.4) must be replaced by

$$F_n(z) = \alpha\Phi^n(z) + o(1), \quad \text{as } n \rightarrow \infty \quad (1.6)$$

(see Theorem 1.1 of [16] for the precise statement).

The primary goal of this note is to find the asymptotics for the derivatives of Faber polynomials at the corner points of $\partial\Omega$. We also consider applications of such asymptotics to Markov-type inequalities for derivatives of polynomials on K .

It is not unexpected that our subject is directly related to the geometric properties of $\partial\Omega$ via the conformal mapping Ψ . Let $z_0 \in \partial\Omega$ be a point such that two analytic arcs of $\partial\Omega$ meet at z_0 and form the angle $\alpha\pi$, $0 < \alpha \leq 2$, as measured in Ω . According to the result of Lehman [10], $\Psi(w)$ allows an asymptotic expansion in the neighborhood of w_0 , where $\Psi(w_0) = z_0$,

$$\Psi(w) - \Psi(w_0) = \begin{cases} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} a_{kl} (w - w_0)^{k+l\alpha}, & \alpha \text{ is irrational,} \\ \sum_{k=0}^{\infty} \sum_{l=1}^q \sum_{m=0}^{[k/p]} a_{klm} (w - w_0)^{k+lp/q} (\log(w - w_0))^m, & \alpha = p/q \text{ is rational.} \end{cases} \quad (1.7)$$

In both cases, the first term of this expansion is given by

$$\Psi(w) - \Psi(w_0) = a_{\alpha} (w - w_0)^{\alpha} + \dots, \quad a_{\alpha} \neq 0 \quad (1.8)$$

(see Theorem 1 of [10] and Section 3.4 of [12] for details). Our main result below gives the asymptotics for the derivatives of Faber polynomials at an “analytic corner.”

THEOREM 1.1. *Let $\partial\Omega$ be rectifiable. Suppose that Ω has the angle $\alpha\pi$, $0 < \alpha \leq 2$, at its boundary point $z_0 = \Psi(w_0)$, which is locally formed by two analytic arcs of $\partial\Omega$. Then*

$$F_n^{(k)}(z_0) = \frac{\alpha k! n^{\alpha k} w_0^n}{(a_{\alpha} w_0^{\alpha})^k \Gamma(\alpha k + 1)} + o(n^{\alpha k}), \quad \text{as } n \rightarrow \infty, \quad (1.9)$$

where $k = 0, 1, 2, \dots$

Note that the appropriate branch of the multiple valued function w^{α} , $0 < \alpha \leq 2$, is defined by the expansion (1.7)-(1.8), together with the associated coefficient a_{α} .

If $k = 0$ then we obtain the asymptotics (1.6) for Faber polynomials themselves (see [16] for a more general result). The case $k = 1$ gives the asymptotics for the first derivative of Faber polynomials, which have applications to Markov-type inequalities for the derivative of polynomials on general sets. The fact that Faber polynomials can be used to show

sharpness of Markov-type inequalities was already observed in the classical paper of Szegő [23]. We develop his ideas and relate our asymptotics to the result of Pommerenke [14] and the conjectures of Erdős [4]-[5].

2. MARKOV INEQUALITIES FOR GENERAL SETS

Define the uniform (sup) norm on K by

$$\|f\|_K := \sup_{z \in K} |f(z)|.$$

The classical Markov inequality for $K = [-1, 1]$ states that

$$\|P'_n\|_{[-1,1]} \leq n^2 \|P_n\|_{[-1,1]}, \quad (2.1)$$

where P_n is a polynomial of degree at most n (cf. Section 5.1 of [1] and [19]). We have equality in (2.1) for the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. On the other hand, Bernstein's inequality for the unit disk D gives

$$\|P'_n\|_D \leq n \|P_n\|_D. \quad (2.2)$$

Obviously, equality holds in (2.2) for $P_n(z) = z^n$. Szegő [23] was apparently the first to explain the nature of difference in the exponents of n in (2.1) and (2.2), using the geometry of sets $[-1, 1]$ and D in the complex plane. He proved that

$$\|P'_n\|_K \leq C(K) n^\alpha \|P_n\|_K, \quad (2.3)$$

where $\alpha\pi$ is the largest angle at $\partial\Omega$, $1 \leq \alpha \leq 2$, and $C(K)$ is independent of $n \in \mathbf{N}$. The exponent α is sharp, as shown by Szegő with the help of Faber polynomials. This also follows from Theorems 1.1 and 2.1, for $k = 1$, which in addition give a lower bound for the constant $C(K)$. Similarly, the asymptotics (1.9) can be used to show the sharpness of inequalities for the derivatives of higher order $k \geq 2$.

A universal Markov-type inequality, for an arbitrary continuum K of capacity $\text{cap}(K)$, was obtained by Pommerenke [14]:

$$\|P'_n\|_K \leq \frac{e n^2}{2 \text{cap}(K)} \|P_n\|_K. \quad (2.4)$$

Erdős conjectured that e could be replaced by 1 in (2.4), so that (2.1) would follow from this general result, as $\text{cap}([-1, 1]) = 1/2$. After Rassias et al.

[18] had noticed that his conjecture needed adjustment, Erdős restated it in the corrected form

$$\|P'_n\|_K \leq \frac{(1+o(1))n^2}{2 \operatorname{cap}(K)} \|P_n\|_K, \quad (2.5)$$

as $n \rightarrow \infty$ (see, e.g., [5]).

Note that if the angle at z_0 is 2π , in the setting of Theorem 1.1, then we have

$$|F'_n(z_0)| = \frac{1+o(1)}{|a_2|} n^2, \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

It is also known that

$$\|F_n\|_K \leq 2, \quad n \in \mathbf{N}, \quad (2.7)$$

for convex K (cf. [13]), so that we can estimate in this case

$$\frac{\|F'_n\|_K}{\|F_n\|_K} \geq \frac{|F'_n(z_0)|}{2} = \frac{1+o(1)}{2|a_2|} n^2, \quad \text{as } n \rightarrow \infty. \quad (2.8)$$

Thus one might try to disprove (2.5) by finding an appropriate set K , such that $|a_2| < \operatorname{cap}(K)$. However, we verified for a number of special cases that

$$|a_2| \geq \operatorname{cap}(K). \quad (2.9)$$

In particular, we have $a_2 = 1/2 = \operatorname{cap}([-1, 1])$ for $K = [-1, 1]$. After the initial version of this paper had been submitted for publication, Kühnau [9] found an elegant proof of (2.9), which is based on a distortion theorem of Löwner [11]. Hence (2.8)-(2.9) show that inequality (2.5) is sharp for sets with outward pointing cusps.

We remark that the convexity of K is not essential in the above argument, because (2.7) can be replaced by the following.

THEOREM 2.1. *If $\partial\Omega$ is a piecewise smooth Jordan curve formed by a finite number of Dini-smooth arcs, then*

$$\limsup_{n \rightarrow \infty} \|F_n\|_K \leq 2. \quad (2.10)$$

A Dini-smooth arc is a Jordan arc with a natural parametrization $z(s)$, such that $z'(s)$ is Dini-continuous, and $z'(s) \neq 0$ for any $s \in [0, l]$ (see, e.g., [12]). Note that the bound 2 in (2.10) cannot be decreased, which is immediate from (1.6) (or from (1.9) with $k = 0$).

3. PROOFS

Proof (Proposition 1.1). Let Ω_r be a domain such that $\Phi(\Omega_r) = \{w : |w| > r\}$, $r > 0$. There exists $r_0 \in (0, 1)$ such that Φ has a conformal extension into Ω_{r_0} . Hence (1.3) is valid for any $z \in \Omega_{r_0}$, and $E_n(z)$ is analytic in Ω_{r_0} . Denote the level curve of Φ by $\gamma_r := \{z : |\Phi(z)| = r\}$, $r > r_0$. Using Cauchy integral formula, we obtain from (1.3) that

$$E_n(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t) dt}{t - z}, \quad z \in \Omega_r, \quad r > r_0,$$

where integration is carried in clockwise direction. It follows by differentiation of (1.3) that

$$F_n^{(k)}(z) = \frac{d^k}{dz^k} (\Phi^n(z)) + \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t) dt}{(t - z)^{k+1}}, \quad z \in \Omega_r, \quad k = 0, 1, 2, \dots \quad (3.1)$$

We can estimate the remainder term for $z \in \Omega_{r'}$, $r < r' < 1$,

$$\left| \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t) dt}{(t - z)^{k+1}} \right| \leq \frac{k!}{2\pi} \frac{l(\gamma_r) r^n}{(\text{dist}(\gamma_r, \gamma_{r'}))^{k+1}}, \quad (3.2)$$

where $l(\gamma_r)$ is the length of γ_r and

$$\text{dist}(\gamma_r, \gamma_{r'}) := \min\{|t - z| : t \in \gamma_r, z \in \gamma_{r'}\}.$$

Thus (1.5) is a consequence of (3.1) and (3.2). \blacksquare

Proof (Theorem 1.1). Using Cauchy formula in (1.3), for a contour $\gamma_r := \{z : |\Phi(z)| = r > 1\}$ and a point $z \in \Omega$ inside γ_r , we have that

$$F_n(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t) dt}{t - z}. \quad (3.3)$$

This well known integral representation of Faber polynomials is valid for any $z \in K$ by analytic continuation (cf. [21]). Thus we obtain from (3.3) that

$$F_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t) dt}{(t - z)^{k+1}} = \frac{k!}{2\pi i} \int_{|w|=r} \frac{w^n \Psi'(w) dw}{(\Psi(w) - z)^{k+1}}, \quad (3.4)$$

where $z \in K$ and $k = 0, 1, 2, \dots$. Since $\partial\Omega$ is rectifiable, $|\Psi'(w)|$ is integrable over $|w| = 1$. Therefore, (3.4) gives that

$$F_n^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{w^n \Psi'(w) dw}{(\Psi(w) - \Psi(w_0))^{k+1}}, \quad z_0 = \Psi(w_0), \quad (3.5)$$

where γ is the contour consisting of the arc $\gamma' := \{w : |w - w_0| = s, |w| > 1\}$ and the arc $\gamma'' := \{w : |w - w_0| \geq s, |w| = 1\}$, for a small but fixed $s > 0$. Using expansion (1.7)-(1.8), we have that (see [10] and Section 3.4 of [12])

$$\Psi(w) - \Psi(w_0) = a_\alpha(w - w_0)^\alpha + g(w - w_0)$$

and

$$\Psi'(w) = \alpha a_\alpha(w - w_0)^{\alpha-1} + g'(w - w_0),$$

for w in a neighborhood of w_0 , $|w| > 1$. The expansion for g starts as follows:

$$g(w - w_0) = \begin{cases} b(w - w_0)^{2\alpha} + \dots, & \alpha < 1, \\ b(w - w_0)^2 \log(w - w_0) + \dots, & \alpha = 1, \\ b(w - w_0)^{1+\alpha} + \dots, & \alpha > 1. \end{cases}$$

Hence

$$\begin{aligned} \frac{\Psi'(w)}{(\Psi(w) - \Psi(w_0))^{k+1}} &= \frac{\alpha}{a_\alpha^k (w - w_0)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right) \quad (3.6) \\ &= \frac{\alpha}{a_\alpha^k w^{\alpha k + 1} (1 - w_0/w)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right) \\ &= \frac{\alpha}{a_\alpha^k w_0^{\alpha k + 1}} \frac{1}{(1 - w_0/w)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right), \end{aligned}$$

where $p < \alpha k + 1$. It follows that

$$\frac{k!}{2\pi i} \int_\gamma \frac{w^n \Psi'(w) dw}{(\Psi(w) - \Psi(w_0))^{k+1}} = \frac{k!}{2\pi i} \left(\int_{\gamma'} + \int_{\gamma''} \right) \frac{w^n \Psi'(w) dw}{(\Psi(w) - \Psi(w_0))^{k+1}}, \quad (3.7)$$

where the integral over γ'' is bounded for all $n \in \mathbf{N}$, as $s \leq |w - w_0| \leq 2$ and $|w| = 1$. Since $1/(1 - w_0/w)^{\alpha k + 1}$ is analytic in $\overline{\mathbf{C}} \setminus [0, w_0]$, we have that

$$\left| \frac{1}{2\pi i} \int_{\gamma'} \frac{w^n dw}{(1 - w_0/w)^{\alpha k + 1}} - \frac{1}{2\pi i} \int_{|w|=r} \frac{w^n dw}{(1 - w_0/w)^{\alpha k + 1}} \right| \leq C(s), \quad (3.8)$$

where $C(s)$ is independent of $n \in \mathbf{N}$. Using the formula for the $(n + 1)$ st coefficient of the Laurent expansion for $1/(1 - w_0/w)^{\alpha k + 1}$ about $w = \infty$, we obtain that

$$\begin{aligned} \frac{1}{2\pi i} \int_{|w|=r} \frac{w^n dw}{(1 - w_0/w)^{\alpha k + 1}} &= \binom{\alpha k + n + 1}{n + 1} w_0^{n+1} \quad (3.9) \\ &\sim \frac{n^{\alpha k}}{\Gamma(\alpha k + 1)} w_0^{n+1}, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The same argument shows that

$$\frac{1}{2\pi i} \int_{|w|=r} \frac{w^n dw}{(1-w_0/w)^p} = O(n^{p-1}) = o(n^{\alpha k}), \quad \text{as } n \rightarrow \infty.$$

Thus we obtain from (3.6)-(3.9) that

$$F_n^{(k)}(z_0) = \frac{\alpha k! n^{\alpha k} w_0^n}{(a_\alpha w_0^\alpha)^k \Gamma(\alpha k + 1)} + o(n^{\alpha k}), \quad \text{as } n \rightarrow \infty,$$

where $k = 0, 1, 2, \dots$. One can deduce more precise information about the error term, by applying similar analysis to the remaining terms of the asymptotic expansion (3.6). \blacksquare

Proof (Theorem 2.1). Observe that Ψ extends to a homeomorphism between $\{w : |w| = 1\}$ and $\partial\Omega$ (see Theorem 2.1 of [12]). Consider the function

$$v(t, \theta) := \arg(\Psi(e^{it}) - \Psi(e^{i\theta})), \quad t \neq \theta. \quad (3.10)$$

Note that $v(t, \theta)$ has a jump discontinuity as a function of t , at $t = \theta$, where $\theta \in [0, 2\pi)$ is fixed. The magnitude of this jump, arising when t passes through θ , is equal to the angle formed by $\partial\Omega$ at $\Psi(e^{i\theta})$, as measured in Ω . Clearly, $v(t, \theta)$ can be defined continuously for $t \neq \theta$. It was proved in [8, Th. 4] that $v(t, \theta)$ is of bounded variation as a function of $t \in [0, 2\pi)$. Hence we have the following integral representation for Faber polynomials

$$F_n(\Psi(e^{i\theta})) = \frac{1}{\pi} \int_0^{2\pi} e^{int} d_t v(t, \theta), \quad 0 \leq \theta < 2\pi, \quad (3.11)$$

which is due to Pommerenke (cf. [13], [15] and [8]).

Let $\delta > 0$ be small. Since $\partial\Omega$ is rectifiable, we have that $\Psi'(e^{it}) \in L^1([0, 2\pi))$, see [12, Th. 6.8]. Thus (3.10) gives that

$$\int_{\theta+\delta}^{\theta+2\pi-\delta} e^{int} d_t v(t, \theta) = \int_{\theta+\delta}^{\theta+2\pi-\delta} e^{int} \Re \left(\frac{e^{it} \Psi'(e^{it})}{\Psi(e^{it}) - \Psi(e^{i\theta})} \right) dt. \quad (3.12)$$

The regular modulus of continuity for a 2π -periodic continuous function f is given by

$$\omega_\infty(f, u) := \sup_{|x-y| \leq u} |f(y) - f(x)|.$$

We also define the L^1 modulus of continuity for a 2π -periodic function $f \in L^1([0, 2\pi))$ by

$$\omega_1(f, u) := \sup_{|h| \leq u} \int_0^{2\pi} |f(x+h) - f(x)| dx.$$

The corresponding L^1 modulus of continuity on $[\theta + \delta, \theta + 2\pi - \delta]$ is denoted by $\omega_1(f, u; \theta)$. Note that

$$\min_{t \in [\theta + \delta/2, \theta + 2\pi - \delta/2]} |\Psi(e^{it}) - \Psi(e^{i\theta})| = c(\delta) > 0.$$

Hence we have for $u \in (0, \delta/2)$

$$\begin{aligned} & \omega_1 \left(\Re \left(\frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it}) - \Psi(e^{i\theta})} \right), u; \theta \right) \leq \omega_1 \left(\frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it}) - \Psi(e^{i\theta})}, u; \theta \right) \\ & \leq \frac{\omega_1(e^{it}\Psi'(e^{it}), u) \max_{t \in [0, 2\pi]} |\Psi(e^{it}) - \Psi(e^{i\theta})|}{(c(\delta))^2} \\ & + \frac{\omega_\infty(\Psi(e^{it}), u) \int_0^{2\pi} |e^{it}\Psi'(e^{it})| dt}{(c(\delta))^2} \\ & \leq \frac{A \omega_1(\Psi'(e^{it}), u) + \omega_\infty(\Psi(e^{it}), u) \int_0^{2\pi} |\Psi'(e^{it})| dt}{(c(\delta))^2}, \end{aligned} \quad (3.13)$$

where A is a positive constant independent of $\theta \in [0, 2\pi)$ and $\delta > 0$. It follows from Section 2.3.7 of [3] and (3.12) that

$$\int_{\theta + \delta}^{\theta + 2\pi - \delta} e^{int} d_t v(t, \theta) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.14)$$

uniformly in $\theta \in [0, 2\pi)$, by a version of the Riemann-Lebesgue lemma.

We show in Lemma 3.1 below that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\theta - \delta}^{\theta + \delta} |d_t v(t, \theta)| \leq 2\pi + \varepsilon, \quad \theta \in [0, 2\pi). \quad (3.15)$$

Combining (3.14), (3.15) and (3.11), we obtain that

$$\limsup_{n \rightarrow \infty} \|F_n\|_K \leq 2 + \frac{\varepsilon}{\pi},$$

which yields (2.10) after letting $\varepsilon \rightarrow 0$. ■

LEMMA 3.1. *Suppose that the assumptions of Theorem 2.1 are satisfied. For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{\theta - \delta}^{\theta + \delta} |d_t v(t, \theta)| \leq 2\pi + \varepsilon, \quad \theta \in [0, 2\pi).$$

Proof. We first note that the above integral expresses the variation of the angle for the secant line through $\Psi(e^{i\theta})$ and $\Psi(e^{it})$, as t runs from $\theta - \delta$ to $\theta + \delta$. This variation is clearly independent of parametrization for the arc

$$\gamma := \{\Psi(e^{it}) : \theta - \delta \leq t \leq \theta + \delta\}.$$

Also, it is well known that variation is an additive function, so that

$$\begin{aligned} \text{Var}_t(v(t, \theta), [\theta - \delta, \theta + \delta]) &= \text{Var}_t(v(t, \theta), [\theta - \delta, \theta]) \\ &+ \text{Var}_t(v(t, \theta), (\theta, \theta + \delta]) + \beta(\theta), \end{aligned} \quad (3.16)$$

where $\beta(\theta)$ is the angle at $\Psi(e^{i\theta})$ as measured in Ω . By choosing $\delta > 0$ sufficiently small, we can assume that γ contains at most one corner point of $\partial\Omega$. If γ is smooth, then $\beta(\theta) = \pi$. Furthermore, for any $\varepsilon > 0$ there is $\delta > 0$, independent of θ , such that

$$\max(\text{Var}_t(v(t, \theta), [\theta - \delta, \theta]), \text{Var}_t(v(t, \theta), (\theta, \theta + \delta])) \leq \varepsilon/2,$$

by Theorem 5 of [8]. This gives that

$$\text{Var}_t(v(t, \theta), [\theta - \delta, \theta + \delta]) \leq \pi + \varepsilon, \quad (3.17)$$

uniformly in θ .

If $\Psi(e^{i\theta})$ is a corner point, then we similarly obtain that

$$\text{Var}_t(v(t, \theta), [\theta - \delta, \theta + \delta]) \leq \beta(\theta) + \varepsilon \leq 2\pi + \varepsilon. \quad (3.18)$$

Consider the remaining case when the corner point is at $\Psi(e^{it_0})$, $t_0 \in (\theta, \theta + \delta)$. Following the same argument as for (3.17), we still have that

$$\text{Var}_t(v(t, \theta), [\theta - \delta, t_0]) \leq \pi + \varepsilon/2, \quad (3.19)$$

for all sufficiently small $\delta > 0$, which are independent of θ . Thus we need to estimate $\text{Var}_t(v(t, \theta), [t_0, \theta + \delta])$. Note that the point $\Psi(e^{i\theta})$ is located outside the arc

$$\gamma_1 := \{\Psi(e^{it}) : t_0 \leq t \leq \theta + \delta\},$$

but it can be arbitrarily close to γ_1 . We now consider a more general variation function

$$h(z) := \text{Var}(\arg(\zeta - z), \zeta \in \gamma_1), \quad z \in \overline{\mathbf{C}}.$$

Let $\zeta_j := \Psi(t_j)$, $j = 0, \dots, k$, where $t_0 < t_1 < \dots < t_k = \theta + \delta$, be a partition of γ_1 . Observe that

$$h_k(z) := \sum_{j=0}^{k-1} |\arg(\zeta_j - z) - \arg(\zeta_{j+1} - z)|$$

is a continuous subharmonic function on $\overline{\mathbf{C}} \setminus \gamma_1$, for any $k \in \mathbf{N}$. By the (generalized) maximum principle for subharmonic functions (cf. Theorems 2.3.1 and 3.6.9 in [17]), we have that

$$h_k(z) \leq \max_{\xi \in \gamma_1} h_k(\xi) \leq \max_{\xi \in \gamma_1} h(\xi), \quad z \in \overline{\mathbf{C}} \setminus \gamma_1.$$

Letting $k \rightarrow \infty$, we obtain that

$$h(z) \leq \max_{\xi \in \gamma_1} h(\xi), \quad z \in \overline{\mathbf{C}} \setminus \gamma_1.$$

Since ξ is now positioned on the smooth arc γ_1 , it follows again that

$$\text{Var}_t(v(t, \theta), [t_0, \theta + \delta]) \leq \max_{\xi \in \gamma_1} \text{Var}(\arg(\zeta - \xi), \zeta \in \gamma_1) \leq \pi + \varepsilon/2,$$

as in (3.17) and (3.19). Combining (3.19) with the above estimate, we have that

$$\text{Var}_t(v(t, \theta), [\theta - \delta, \theta + \delta]) \leq 2\pi + \varepsilon$$

in this remaining case too, so that the Lemma is proved. ▀

Acknowledgement. The author would like to thank Professor D. Gaier and the referee for valuable suggestions, and Professor R. Kühnau for communicating his nice proof of (2.9).

REFERENCES

1. P. Borwein and T. Erdélyi, “Polynomials and Polynomial Inequalities,” Springer-Verlag, New York, 1995.
2. J. H. Curtiss, Faber polynomials and the Faber series, *Amer. Math. Monthly* **78** (1971), 577–596.
3. R. E. Edwards, “Fourier Series, a Modern Introduction,” vol. 1, Springer-Verlag, New York, 1979.
4. P. Erdős, Problem #564, *Colloq. Math.* **15** (1966), 320.
5. P. Erdős, Some of my favourite unsolved problems, in “Tribute to Paul Erdős” (A. Baker, B. Bollobás and A. Hajnal, eds.), pp. 467–478, Cambridge Univ. Press, Cambridge, 1990.

6. D. Gaier, "Lectures on Complex Approximation," Birkhäuser, Boston, 1987.
7. D. Gaier, On the decrease of Faber polynomials in domains with piecewise analytic boundary, *Analysis* **21** (2001), 219-229.
8. D. Gaier, The Faber operator and its boundedness, *J. Approx. Theory* **101** (1999), 265-277.
9. R. Kühnau, personal communication.
10. R. S. Lehman, Development of the mapping function at an analytic corner, *Pacific J. Math.* **7** (1957), 1437-1449.
11. K. Löwner, Über Extremumsätze bei der konformen Abbildung des Äusseren des Einheitskreises, *Math. Z.* **3** (1919), 65-77.
12. Ch. Pommerenke, "Boundary Behaviour of Conformal Maps," Springer-Verlag, Berlin, 1992.
13. Ch. Pommerenke, Über die Faberschen Polynome schlichter Funktionen, *Math. Z.* **85** (1964), 197-208.
14. Ch. Pommerenke, On the derivative of a polynomial, *Mich. Math. J.* **6** (1959), 373-375.
15. Ch. Pommerenke, Konforme Abbildung und Fekete-Punkte, *Math. Z.* **89** (1965), 422-438.
16. I. E. Pritsker, On the local asymptotics of Faber polynomials, *Proc. Amer. Math. Soc.* **127** (1999), 2953-2960.
17. T. Ransford, "Potential Theory in the Complex Plane," Cambridge University Press, Cambridge, 1995.
18. G. M. Rassias, J. M. Rassias and Th. M. Rassias, A counter-example to a conjecture of P. Erdős, *Proc. Japan Acad. Sci.* **53** (1977), 119-121.
19. Q. I. Rahman and G. Schmeisser, "Les Inégalités de Markoff et de Bernstein," Presses de l'Univ. de Montréal, Montréal, 1983.
20. V. I. Smirnov and N. A. Lebedev, "Functions of a Complex Variable: Constructive Theory," MIT Press, Cambridge, 1968.
21. P. K. Suetin, "Series of Faber Polynomials," Gordon and Breach Sci. Publ., Amsterdam, 1998.
22. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc., Providence, 1975.
23. G. Szegő, Ueber einen Satz von A. Markoff, *Math. Z.* **23** (1925), 45-61.