

IA. INTRODUCTION TO THE HOMEOMORPHISM PROBLEM

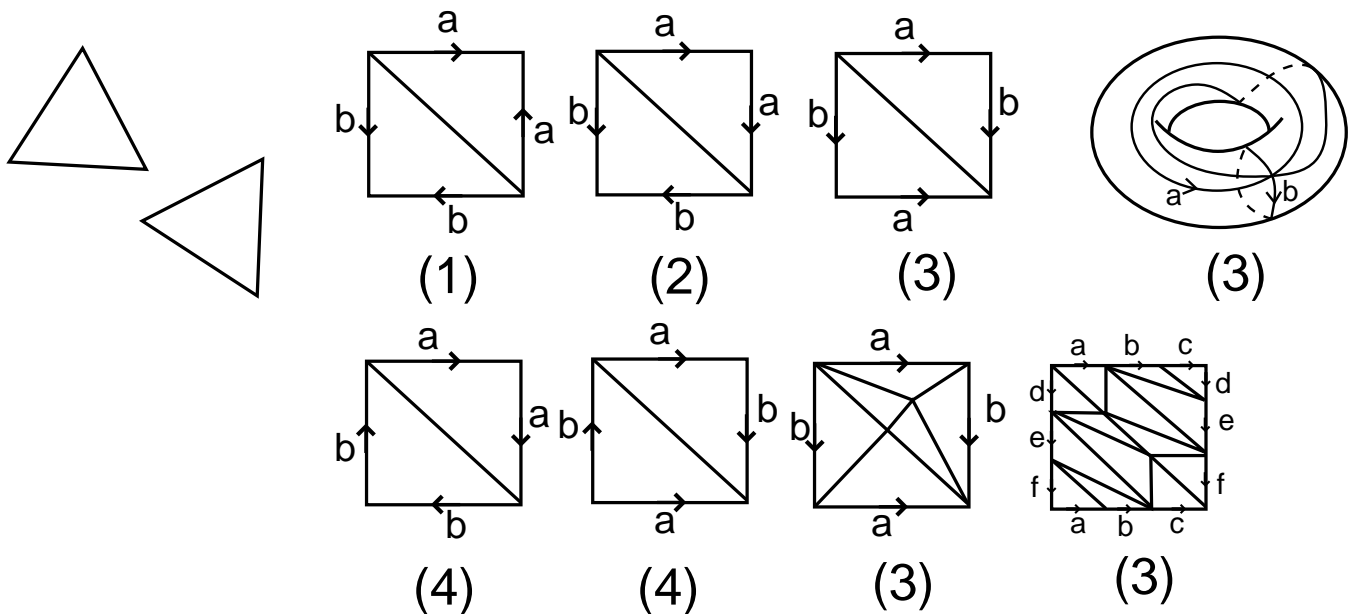
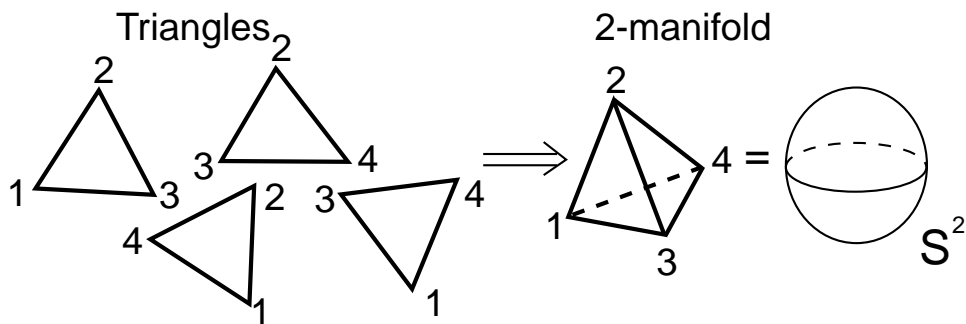
Abstract: This lecture will introduce the basic structure theorems for 3-manifolds (Prime Decomposition, JSJ Decomposition and Thurston's Geometrization Conjecture) and outline the solution to the Homeomorphism Problem (and Classification) for 3-manifolds.

HOMEOMORPHISM PROBLEM: Given the 3-manifolds M and M' to determine if M is homeomorphic to M' .

CLASSIFICATION OF 3-MANIFOLDS: Determine a complete list $M_1, M_2, \dots, M_i, \dots$ of 3-manifolds without duplications (along with an algorithm that decides for any given 3-manifold precisely where it is on the list).

CLASSIFICATION OF 2-MANIFOLDS:

A 2-manifold is given by a triangulation [T.Radó, 1925].



Euler Characteristic of $X \equiv \chi(X) = \text{faces} - \text{edges} + \text{vertices} = f - e + v = -f/2 + v$

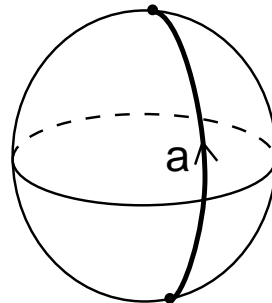
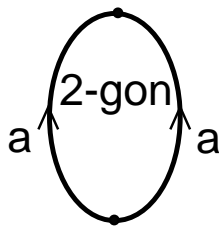
Theorem (Dehn/Heegaard, 1907). *2-manifolds can be classified.*

Euler Char.	Orientable	Non-orientable
$\chi = 2$	S^2	
$\chi = 1$		$\mathbb{R}P^2$
$\chi = 0$	$T^2 = S^1 \times S^1$	$\mathbb{R}P^2 \# \mathbb{R}P^2$
$\chi = -1$		$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$
$\chi = -2$	$T^2 \# T^2$	$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$
\vdots	\vdots	\vdots
\vdots	\vdots	\vdots
$\chi = -2n$	$\underbrace{T^2 \# \dots \# T^2}_{n\text{-terms}}$	$\underbrace{\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2}_{2n\text{-terms}}$
\vdots	\vdots	\vdots

Theorem. (HOMEOMORPHISM PROBLEM)
Given 2-manifolds F and F' , they are homeomorphic if and only if they have the same orientability class and the same Euler characteristic.

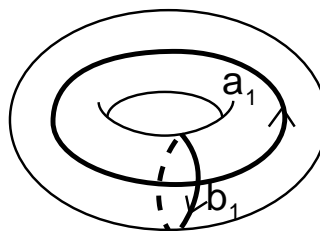
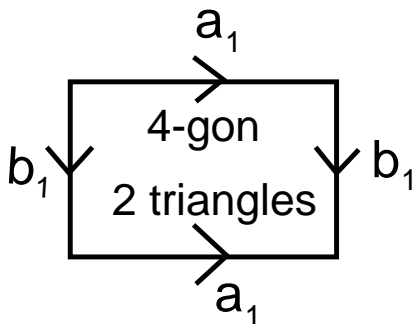
CLASSIFICATION OF ORIENTABLE 2-MANIFOLDS

Euler Characteristic of $(X) = \chi(X) = \text{faces} - \text{edges} + \text{vertices}$
 $= f - e + v = -f/2 + v$



S^2

genus = 0
 $\chi = 2$



$T = S^1 \times S^1$

genus = 1
 $\chi = 0$

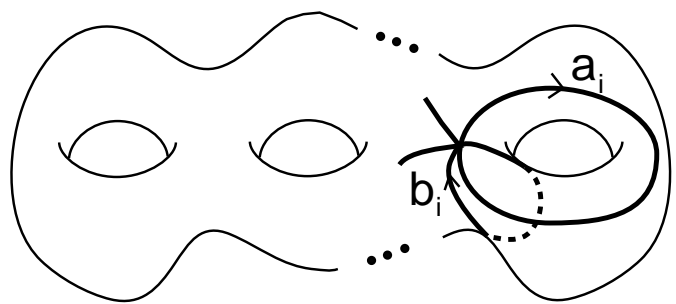
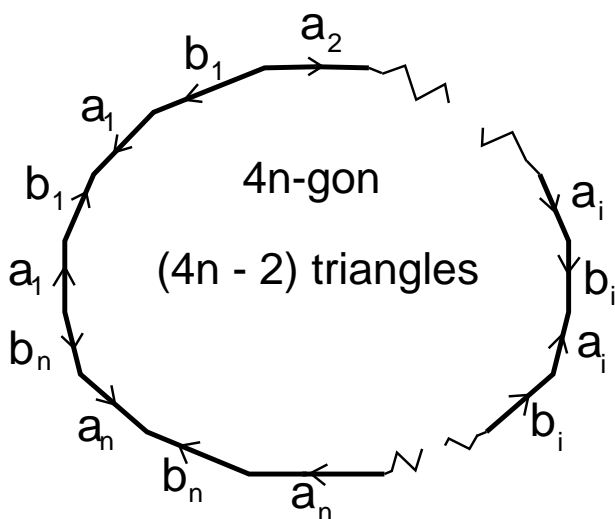
$a_1 b_1 a_1^{-1} b_1^{-1}$

\vdots

\vdots

\vdots

\vdots



genus = n

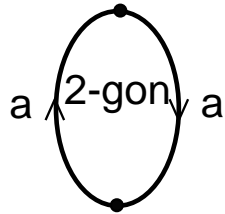
$\chi = 2 - 2n$

$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_i b_i a_i^{-1} b_i^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}$

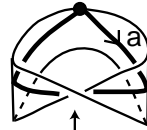
CLASSIFICATION OF NONORIENTABLE 2-MANIFOLDS

Euler Characteristic of $(X) = \chi(X) = \text{faces} - \text{edges} + \text{vertices}$

$$= f - e + v = -f/2 + v$$

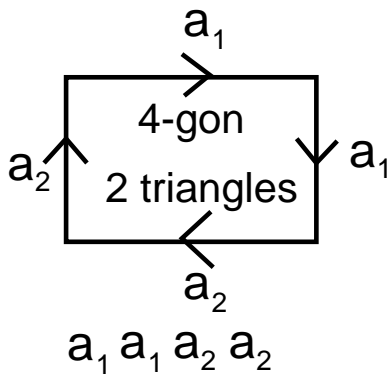


Moebius Band
crosscap

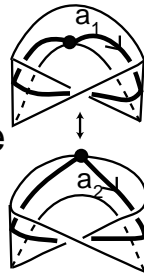


genus = 1

$\mathbb{R}P^2$
 $\chi = 1$



Klein Bottle



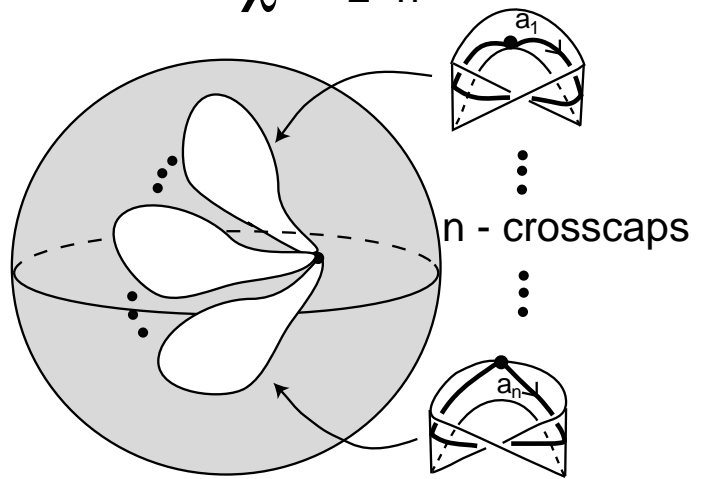
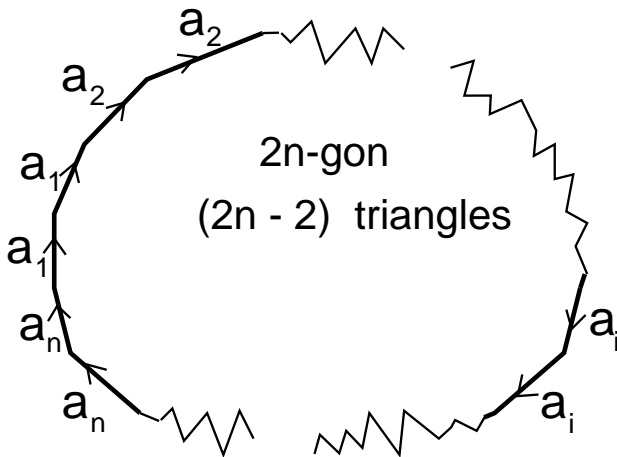
genus = 2

$\chi = 0$

⋮

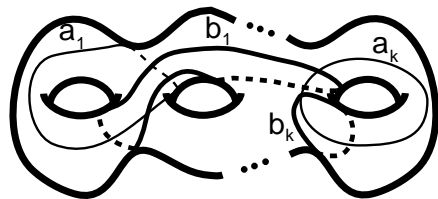
genus = n

$\chi = 2 - n$

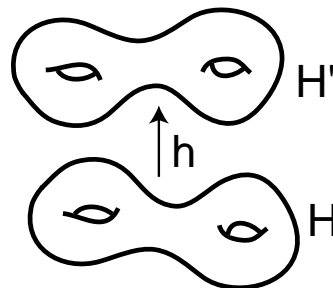


HOMEOMORPHISM PROBLEM FOR 3-MANIFOLDS

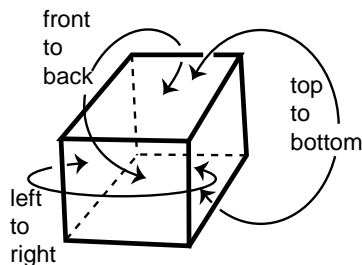
3-manifolds will be given by triangulations



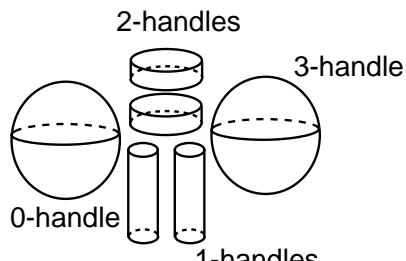
Heegaard Diagram



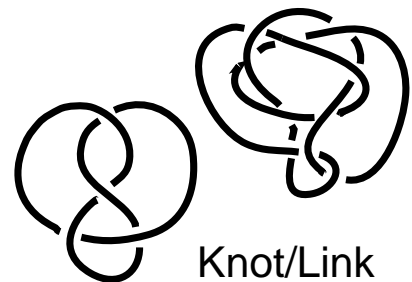
Heegaard Splitting



Cell Decomposition



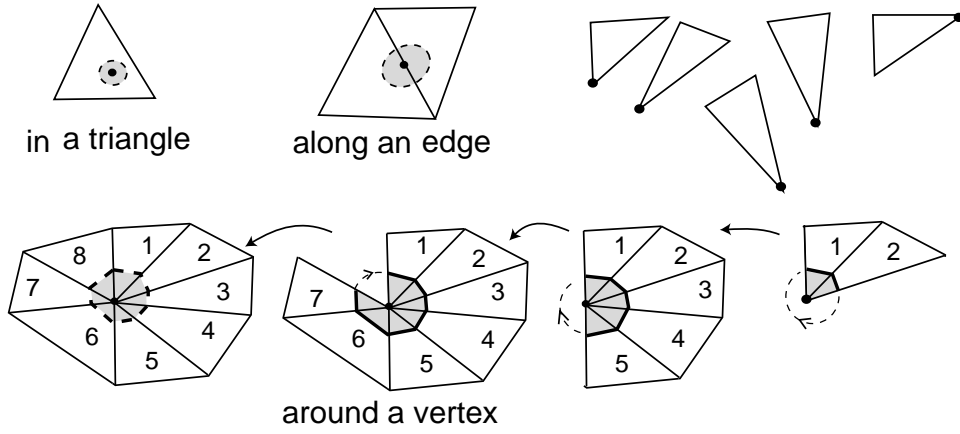
Handle
Decomposition



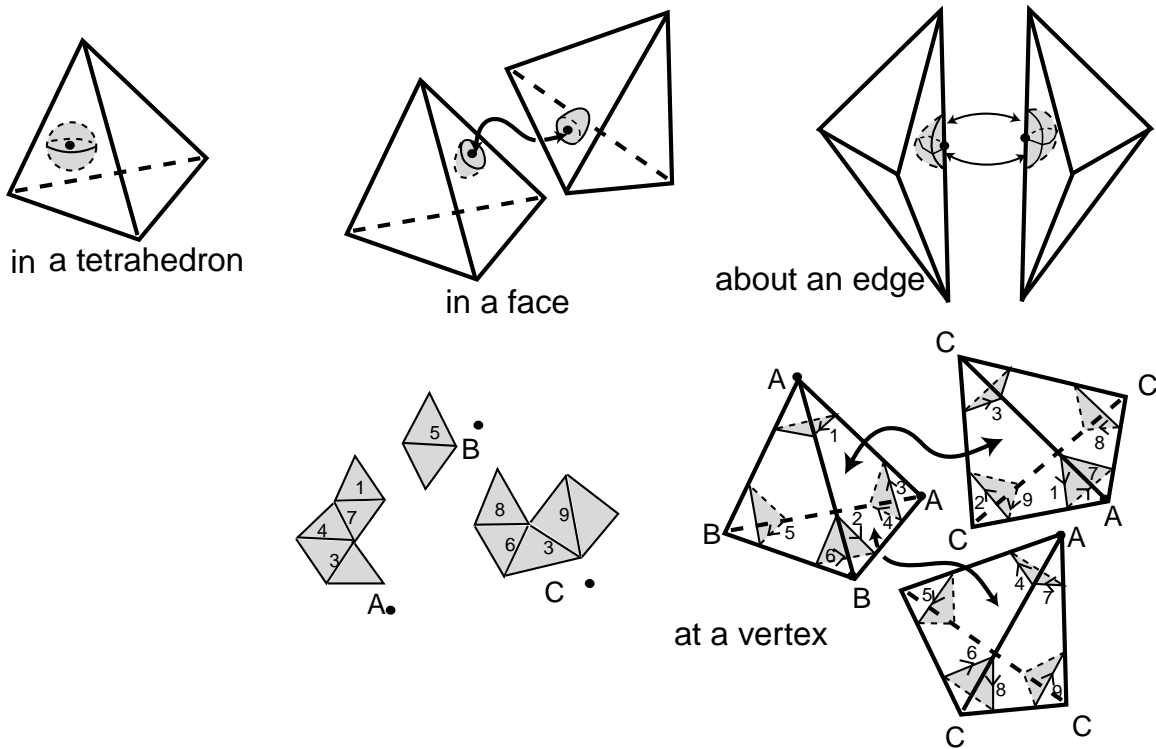
Knot/Link
Projection

Theorem (Moise,1952;Bing,1959). *3-manifolds can be triangulated. Furthermore, any two triangulations of the same 3-manifold have isomorphic subdivisions.*

2-dimensional neighborhoods



3-dimensional neighborhoods



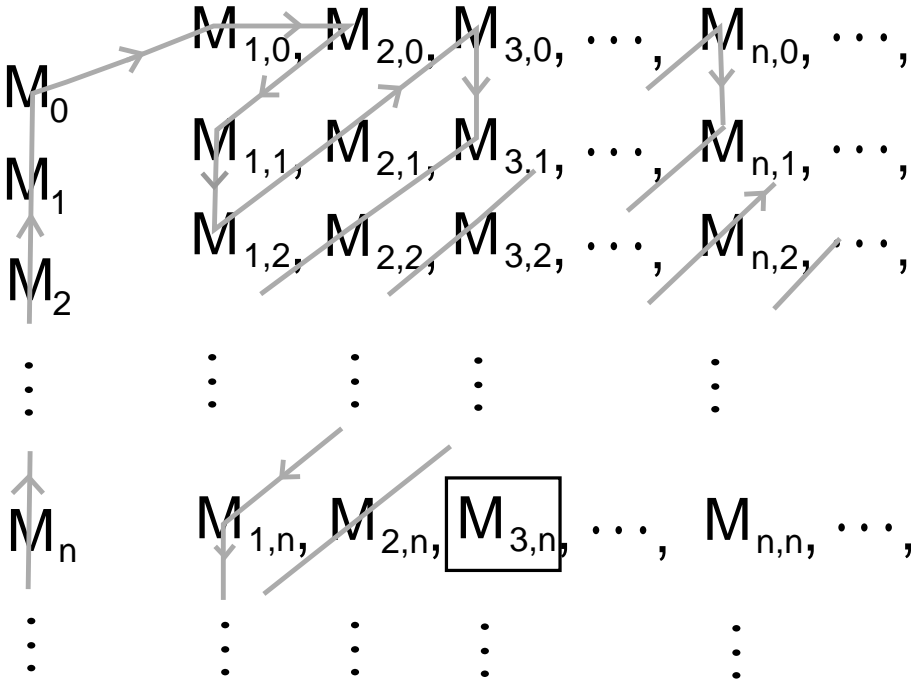
Theorem. (*Haken Cheapological Trick*) *The Homeomorphism Problem for 3-manifolds is equivalent to the (weak) Classification Problem for 3-manifolds.*

- Homeomorphism Problem \Rightarrow Classification.

$M_1, M_2, M_3, \dots, M_n, \dots$

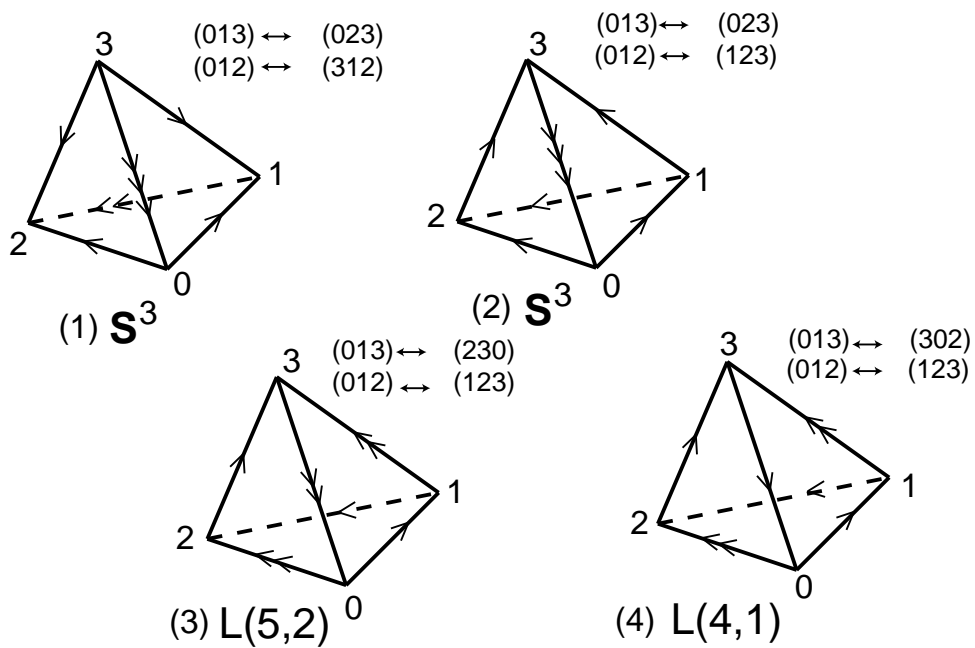
- (weak) Classification \Rightarrow Homeomorphism Problem.

Given: $M = M_0$



EXAMPLES: closed 3-manifolds from one tetrahedron

A. One-tetrahedron



REMARK: The Euler Characteristic of a closed 3-manifold is **zero**.

$$\chi = -t + f - e + v = 0 \Rightarrow t = e - v$$

STEP 1.

 M M' **Prime Decomposition**

$$M = M_1 \# \cdots \# M_n$$

$$M' = M'_1 \# \cdots \# M'_n$$

$$N = M_i$$

STEP 2.

$$N' = M'_j$$

 N N' **JSJ Decomposition** (N, Σ) (N', Σ')

Possibilities:

$$\Sigma = N$$

$$\emptyset \neq \Sigma \neq N$$

$$\Sigma = \emptyset$$

PRIME DECOMPOSITION

DEF: Suppose the 2-sphere Σ separates M into two components with closures M_1 and M_2 . We write

$$M = M_1 \# M_2$$

and say M is a **connected sum** of M_1 and M_2 .

DEF: The 3-manifold M is **prime** if $M \neq S^3$ and $M = M_1 \# M_2$ implies either $M_1 = S^3$ or $M_2 = S^3$.

DEF: A 3-manifold M is **irreducible** if every 2-sphere embedded in M bounds a 3-cell in M .

OBSERVATION: irreducible \Rightarrow prime; prime and not $S^2 \times S^1 \Rightarrow$ irreducible.

Theorem (Kneser,1927). *(Existence) A compact 3–manifold $M \neq S^3$ admits a connected sum decomposition, $M = M_1 \# \dots \# M_n$, where each M_i is prime.*

Theorem (Milnor,1962). *(Uniqueness) If $M = M_1 \# \dots \# M_n$ is a prime decomposition of M , then the factors are unique up to order and homeomorphism.*

Theorem (Jaco-Rubinstein, 1998). (*Construction of Prime Decomposition*) Given a 3–manifold M with triangulation \mathcal{T} , there is an algorithm that constructs a prime decomposition $M = M_1 \# \dots \# M_n$ of M or concludes $M = S^3$.

REMARKS:

1. The algorithm gives M_i by a triangulation \mathcal{T}_i .
2. The triangulation \mathcal{T}_i of M_i is 0–efficient or $M_i = \mathbb{R}P^3$ or $S^2 \times S^1$.
3. $\sum |\mathcal{T}_i| \leq |\mathcal{T}|$ with equality iff \mathcal{T} is 0–efficient.

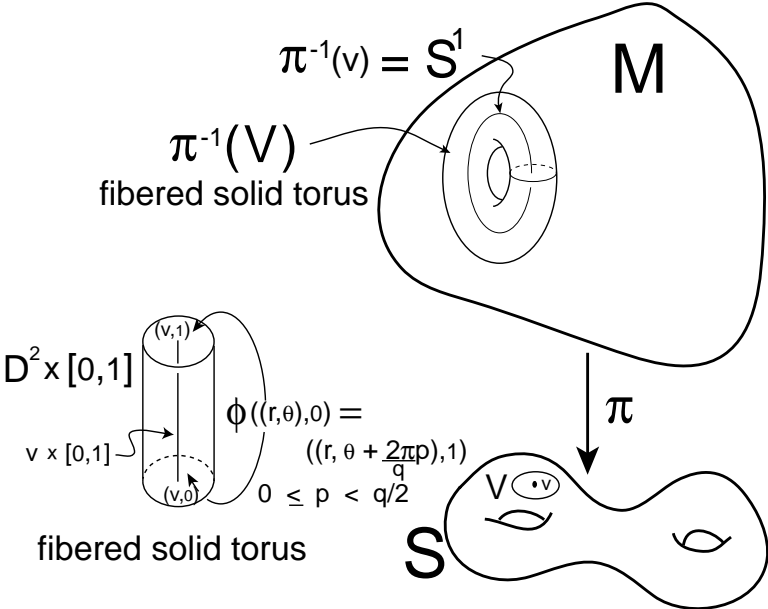
4. We actually have:

$$M = p(\mathbb{R}P^3) \# q(S^2 \times S^1) \# M_1 \# \dots \# M_r,$$

where M_i has a 0–efficient triangulation and $M_i \neq S^3$, $n = p + q + r$.

5. **(Other)** Constructions of Irreducible Decompositions: [Haken, 1961]; [Jaco-Tollefson, 1989]; [Jaco-Rubinstein, 1990]

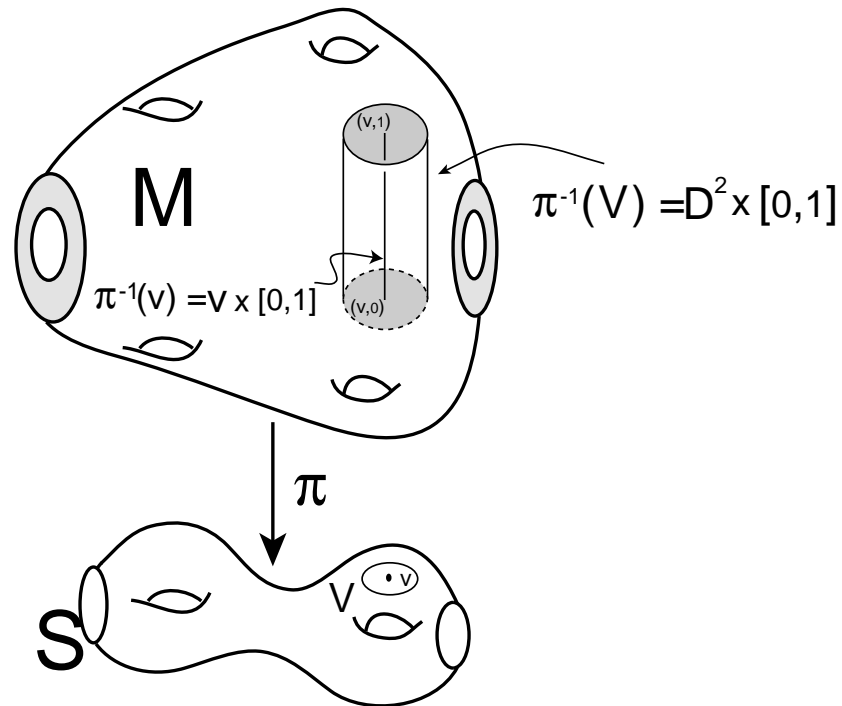
A **Seifert fibred 3–manifold** is a generalized S^1 –bundle over a surface.



Theorem (H. Seifert, 1933). *There is a complete set of invariants (Seifert invariants) for Seifert fibred spaces.*

REMARK: Given two Seifert fibred spaces via their Seifert invariants, it can be decided if they are homeomorphic.

I-Bundle



Fact: An I -bundle is completely determined by its base (recall that we are assuming all 3-manifolds are orientable)

DEF: A 2–sphere embedded in a 3–manifold M is **essential** if it does not bound a 3–cell in M .

An embedded surface $S (\neq S^2)$ is **essential** in M if it is incompressible, ∂ –incompressible and not parallel into ∂M .

incompressible iff $\pi_1(S) \rightarrow \pi_1(M)$ is injective

∂ –incompressible iff $\pi_1(S, \partial S) \rightarrow \pi_1(M, \partial M)$ is injective

DEF: An irreducible 3–manifold is said to be a **Haken 3–manifold** if it contains an essential surface.

DEF: A 3–manifold is said to be **atoridal** if it has no essential embedded torus.

JSJ DECOMPOSITION

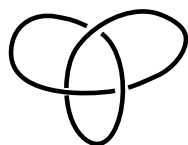
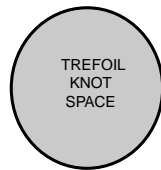
Theorem (Jaco-Shalen, 1979; Johannson, 1979).

N a closed, irreducible 3-manifold, then there is a unique (up to isotopy) submanifold Σ of N so that each component of Σ is Seifert fibred and each component of $N \setminus \overset{\circ}{\Sigma}$ is atoroidal.

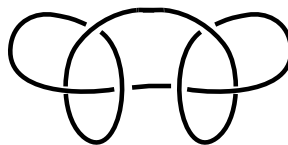
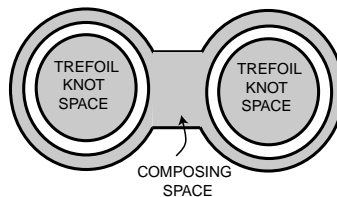
- There is a version for compact, irreducible 3-manifolds with incompressible boundary.

The submanifold Σ is called the **Characteristic Submanifold** of N and we write the pair as (N, Σ) .

$$\begin{array}{ccc} \Sigma = N & \emptyset \neq \Sigma \neq N & \Sigma = \emptyset \\ \square = \Sigma & \square = N \setminus \overset{\circ}{\Sigma} & \end{array}$$



TREFOIL KNOT



SQUARE KNOT

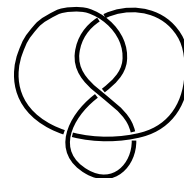
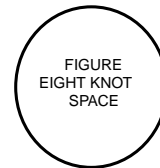


FIGURE EIGHT KNOT

Theorem (Jaco-Tollefson,1989; Jaco-Letscher-Rubinstein,1998). *Given a Haken 3–manifold N , there is an algorithm to construct the JSJ-Decomposition (N, Σ) of N . Furthermore, the algorithm produces the Seifert invariants for each component of Σ .*

REMARKS:

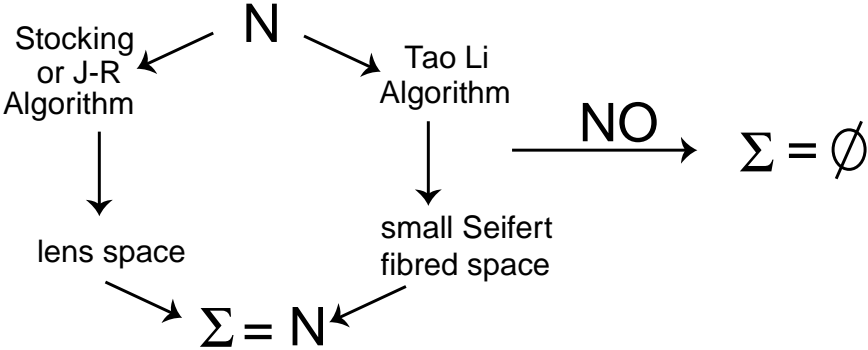
1. A component of $N \setminus \overset{\circ}{\Sigma}$ may be $S^1 \times S^1 \times I$; the algorithm recognizes this.
2. **Corollary.** *Given a Haken 3–manifold it can be decided if it is Seifert fibered; furthermore, if it is the algorithm gives its Seifert invariants.*
3. *We are given an N that is known to be irreducible. We have:*

Theorem (Jaco-Oertel,1981). *Given an irreducible 3–manifold N , it can be decided if N is a Haken-3–manifold.*

For N irreducible, we have: $\Sigma = N$; $\Sigma = \emptyset$;
 $N \neq \Sigma \neq \emptyset$ or N is closed, irreducible and
 atoridal and we don't know if $\Sigma = N$ or $\Sigma = \emptyset$.

N closed, irreducible and atoridal

QUESTION: Is N Seifert fibred?



Theorem (Stocking, 1998; Jaco-Rubinstein, 2001). *Given a closed, irreducible 3-manifold N , it can be decided if N is a lens space and the algorithm produces its Seifert invariants.*

Theorem (Tao Li, 2002). *Given a closed, irreducible 3-manifold N it can be decided if N is a small Seifert fibred space and the algorithm produces its Seifert invariants.*

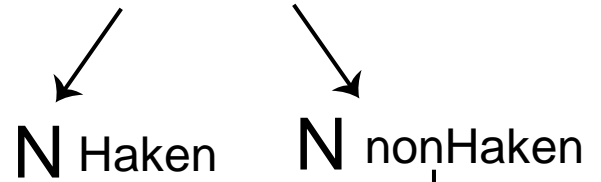
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(N, Σ) JSJ - DECOMPOSITION

$\Sigma = N$

$\emptyset \neq \Sigma \neq N$

$\Sigma = \emptyset$



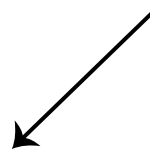
Seifert fibred
or
I-Bundle

1) components of Σ are Seifert fibred or I-bundle

$\partial N \neq \emptyset$

$\partial N = \emptyset$

Thurston's Geometrization Conjecture



2) components of $N \setminus \overset{\circ}{\Sigma}$ are atoroidal, Haken-manifolds

Thurston's Hyperbolization Theorem

Perelman's Claim



Apply Haken's Algorithm

N (hyperbolic)



Apply Sela Algorithm

HOMEOMORPHISM PROBLEM FOR HAKEN-MANIFOLDS

Theorem (Haken,1962; Hemion,1979; Matveev,1995). *Given a Haken 3–manifold N and a 3–manifold N' , it can be decided if N is homeomorphic to N' .*

REMARK: An irreducible 3–manifold M with nonempty boundary is a Haken 3–manifold.

KNOT PROBLEM: Given knots K and K' it can be decided if K is equivalent to K' .

REMARK: [Haken, 1962] In particular, given a knot K , it can be decided if K is the trivial knot (the unknot).

HOMEOMORPHISM PROBLEM FOR HYPERBOLIC 3-MANIFOLDS

DEF: A 3-manifold is **hyperbolic** if its interior has a complete, locally homogeneous, Riemannian metric modelled on \mathbb{H}^3 , hyperbolic 3-space.

Theorem (Sela,1995). *Given closed hyperbolic 3-manifolds N and N' , it can be decided if N is homeomorphic to N' .*

REMARKS:

1. Sela gives an algorithm for deciding the isomorphism problem for certain hyperbolic groups.
2. The application of Sela's Algorithm to hyperbolic 3-manifolds uses the Mostow Rigidity Theorem [**Mostow,1973**] - closed, homotopy equivalent, hyperbolic 3-manifolds are homeomorphic.

THURSTON'S GEOMETRIZATION CONJECTURE [Thurston,1982]: Suppose N is a prime 3-manifold (orientable and incompressible boundary), (N, Σ) is its JSJ-Decomposition. Then the components of both Σ and $\overline{N \setminus \Sigma}$ are geometric.

REMARKS:

1. **[Thurston' Hyperbolization Theorem, 1979]** Geometrization Conjecture is TRUE for Haken 3-manifolds.
2. Geometrization Conjecture \Rightarrow Poincaré Conjecture.
3. Geometrization Conjecture \Rightarrow Spherical Space Form Problem.
4. Geometrization Conjecture \Rightarrow N nonHaken and infinite fundamental group, then N is Seifert fibred or hyperbolic.
5. Geometrization Conjecture \Rightarrow Classification of 3-manifolds.

CLAIM: [G.Perelman,2002-03]
Thurston's Geometrization Con-
jecture is true.

Theorem (Markov,1958). *The Homeomorphism Problem is **unsolvable** for triangulated n -manifolds, $n \geq 4$.*