

# IB. THE CLASSIFICATION OF MANIFOLDS

**Abstract:** This lecture will give a slightly new approach to the classification of 2–manifolds and a new proof of the topological invariance of Euler characteristic for 2–manifolds; we will discuss difficulties of extending these methods to the classification of 3–manifolds; and show the impossibility of classifying  $n$ –manifolds for  $n \geq 4$ .

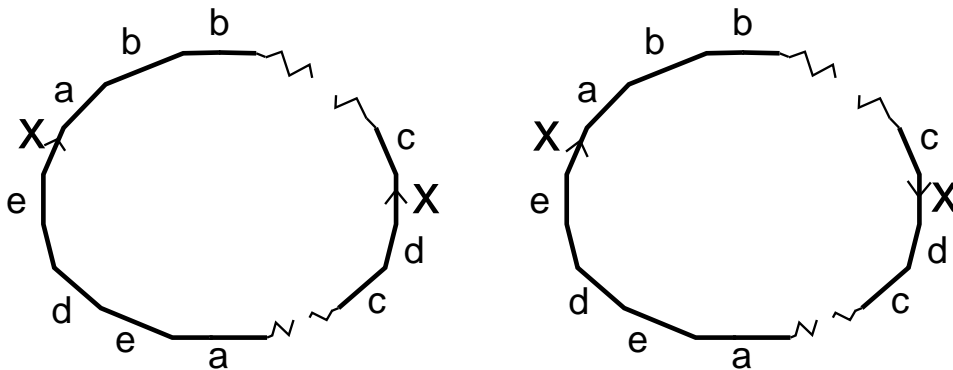
**CLASSIFICATION OF 2-MANIFOLDS:** Provide a complete list  $M_1, M_2, \dots, M_i, \dots$  of 2-manifolds, without duplications, along with an algorithm that decides for any given 2-manifold precisely where it is on the list.

A 2-manifold is to be given by a triangulation.

The **LIST** will be given as a list of regular, planar polygons with boundary edges identified in pairs in some canonical way; there will be numerous, equally as good, canonical triangulations for each such presentation.

## OBSERVE:

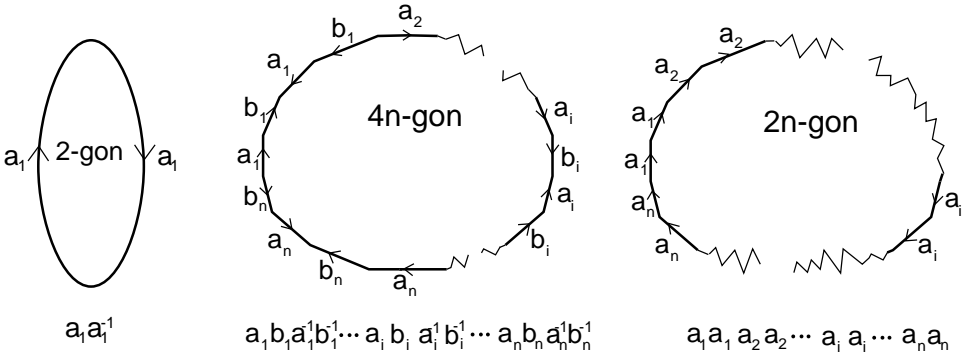
1. The identification space obtained from a regular, planar polygon with boundary edges identified in pairs is a 2-manifold.



2. A regular  $p$ -gon in the plane can be triangulated without adding vertices, using  $(p - 2)$  triangles. Such a triangulation is a *minimal* triangulation of the regular  $p$ -gon.

## Discuss Catalan Numbers

**Definition.** A planar polygon giving a 2-manifold is said to be in **canonical form** if it has edge identifications one of the following:



**Definition.** A triangulation of a 2-manifold obtained from a minimal triangulation of a canonical form is called a **canonical triangulation** of the 2-manifold. Such a triangulation has only one vertex.

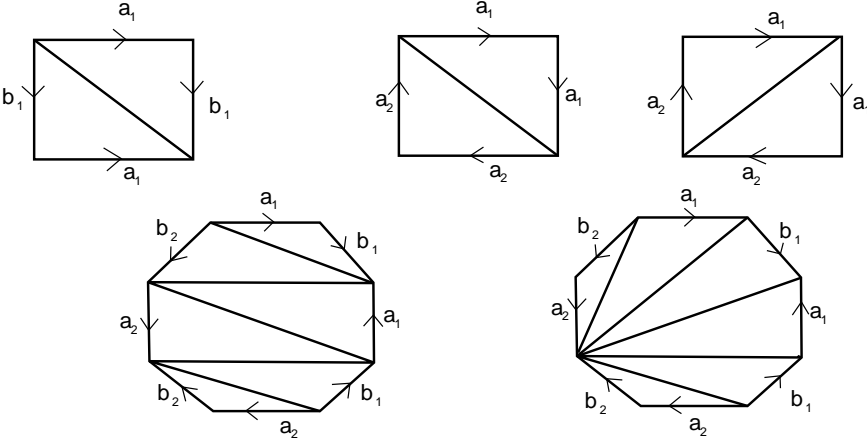
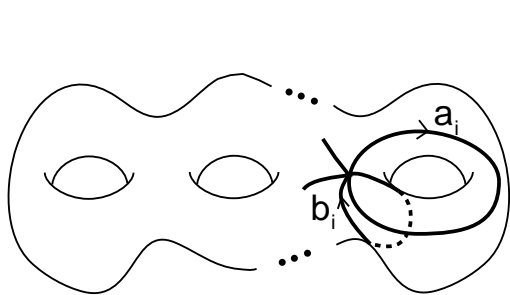
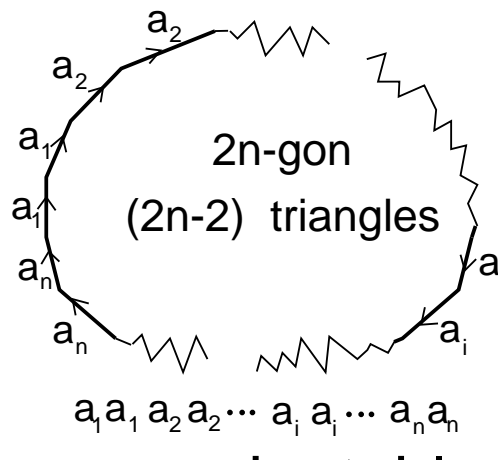
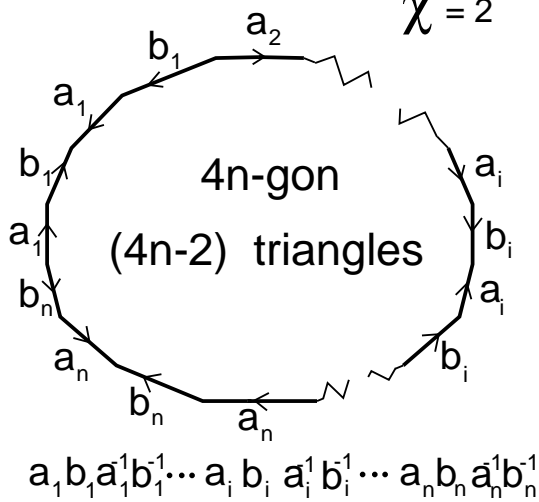
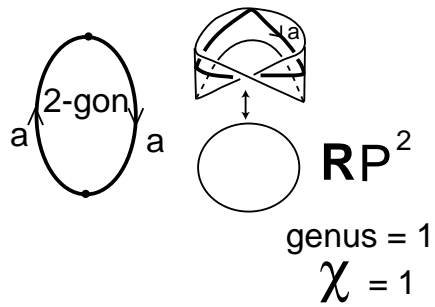
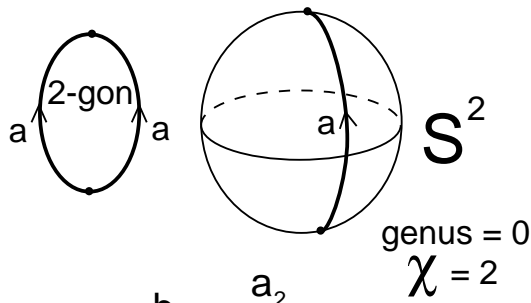


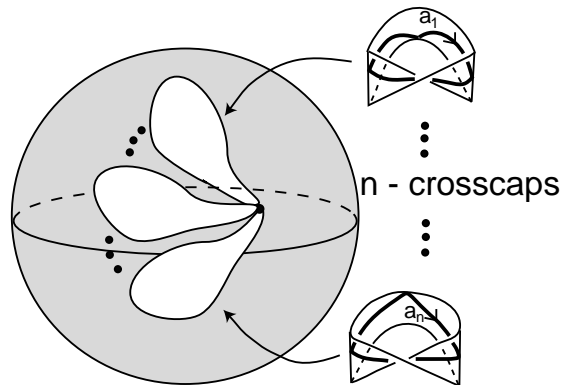
FIGURE: Examples of Canonical Triangulations

# THE LIST

## CANONICAL FORM FOR 2-MANIFOLDS



genus = n  
 $\chi = 2 - 2n$



genus = n  
 $\chi = 2 - n$

## Plan for Classification and Proof of Invariance of Euler Characteristic

- ▷ Given a 2-manifold  $F$  via a triangulation  $\mathcal{T}$
- ▷ Modify triangulation to a one-vertex triangulation without changing Euler characteristic or have  $F = S^2$  or  $\mathbb{R}P^2$
- ▷ Surface can be given as a triangulated planar polygon having all vertices in boundary.
- ▷ Can modify to a minimal triangulation of canonical planar polygon without changing Euler characteristic
- ▷ Every 2-manifold appears on LIST
- ▷ The LIST is without repeats
- ▷ Euler characteristic is a topological invariant

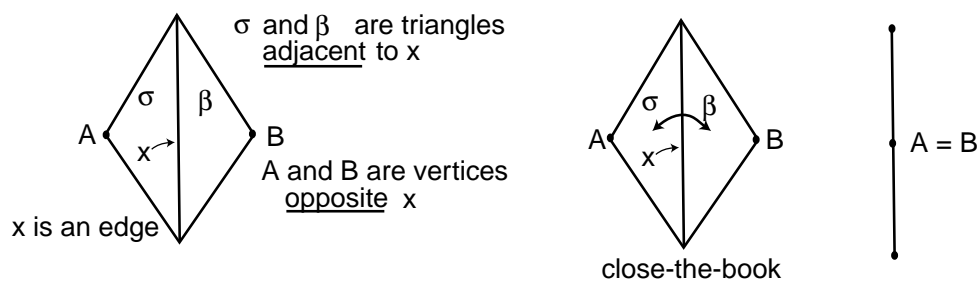
# Modifying Triangulations of 2-manifolds

**STEP 1.** Modify triangulation to a triangulation having only one-vertex.

**Proposition.** Any triangulation of a 2-manifold can be modified to a one-vertex triangulation or it can be shown that the 2-manifold is  $S^2$  or  $\mathbb{R}P^2$ .

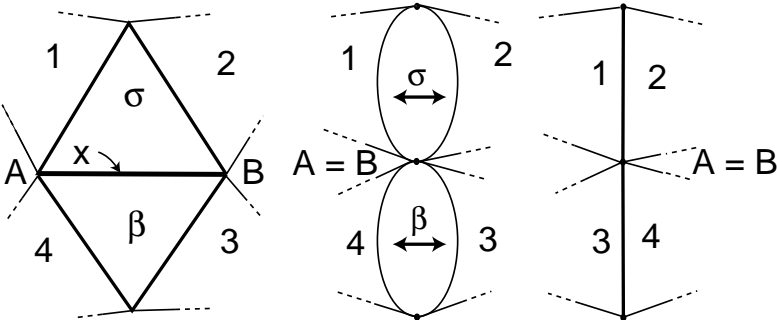
**Remark:** We use a method referred to as “close-the-book”; note that the usual method used in text books loses sight of the triangulation.

**Definition.** triangles adjacent to an edge; vertex opposite an edge



CRUSH AN EDGE (Same as closing the book on the opposite diagonal)

If  $x$  is an edge with distinct vertices  $A$  and  $B$ , we “crush the edge”  $x$  as follows:



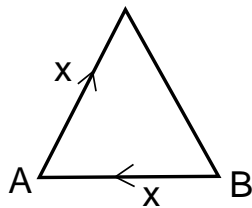
**Lemma.** *Suppose  $\mathcal{T}$  is a triangulation of the 2-manifold  $F$  and  $\mathcal{T}$  has more than one vertex. Then for any edge  $x$  having distinct vertices, we can crush the edge  $x$  and get a new triangulation  $\mathcal{T}'$  of  $F$  having two fewer faces or we have  $F = S^2$  or  $F = \mathbb{R}P^2$  and  $\mathcal{T}$  has just two faces.*

**Proof:**

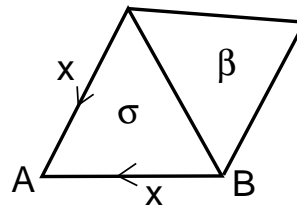
$\mathcal{T}$  has distinct vertices A and B,  $x$  is an edge between A and B

Case 1. There is only one triangle adjacent to  $x$ .

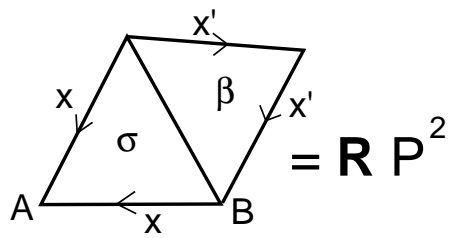
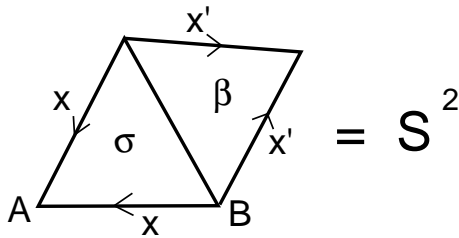
ONLY ONE TRIANGLE ADJACENT TO EDGE



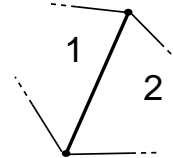
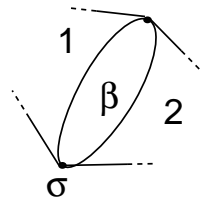
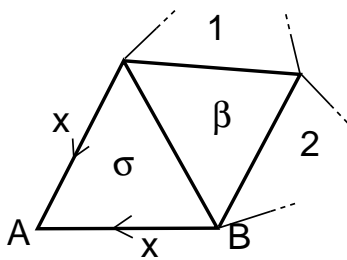
Not Possible ( $A \neq B$ )



only possibility

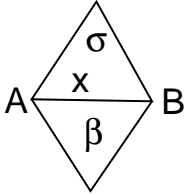


edges of  $\beta$  identified

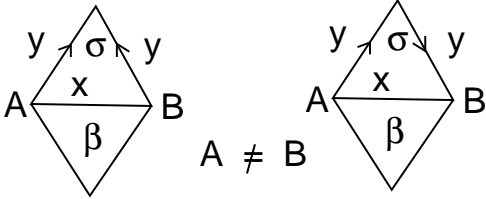


edges of  $\beta$  not identified

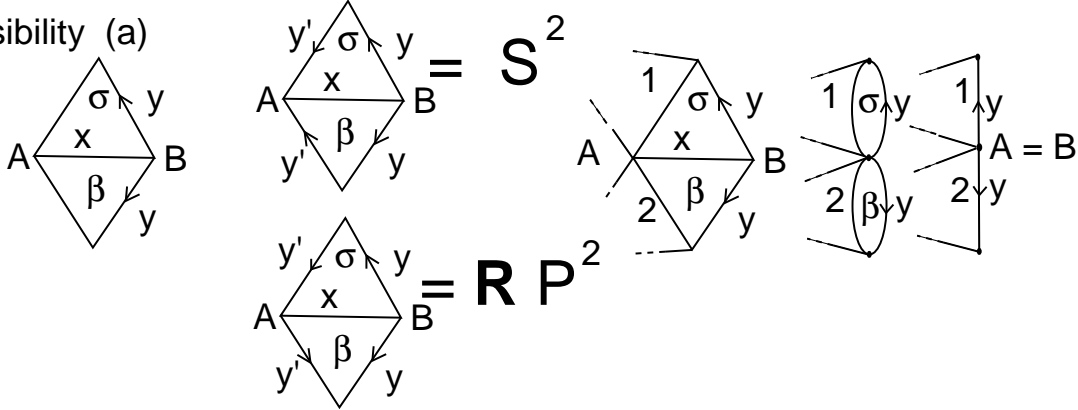
Case 2. There are distinct triangles adjacent to the edge  $x$



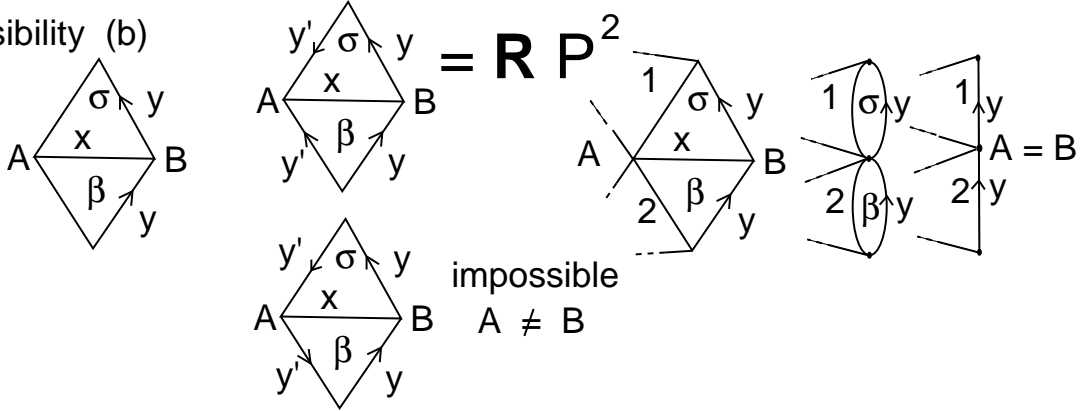
Observe 1. Not Possible



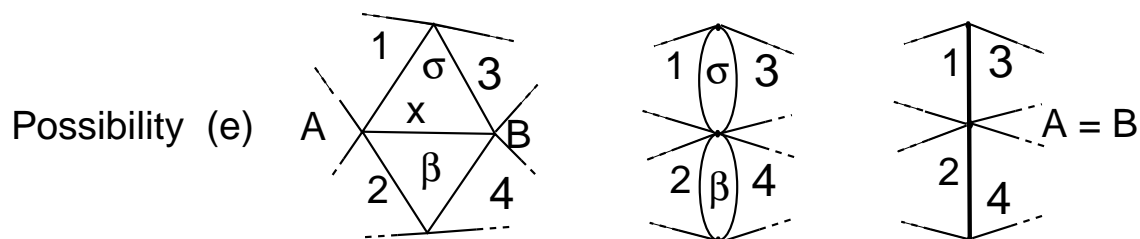
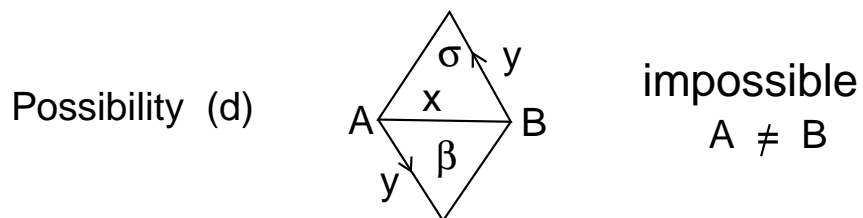
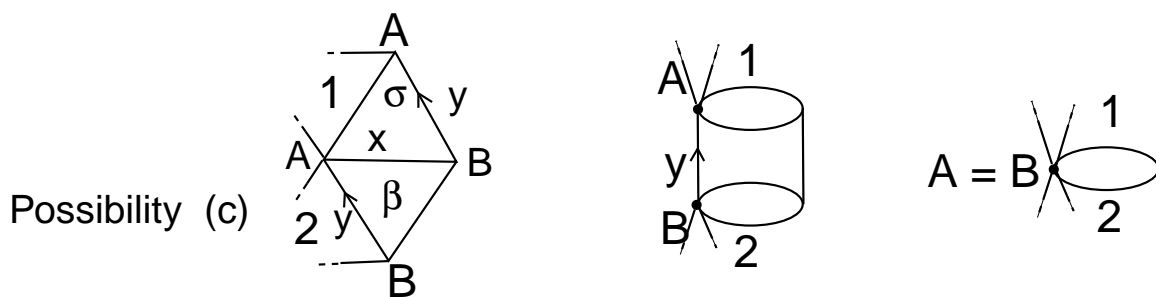
Possibility (a)



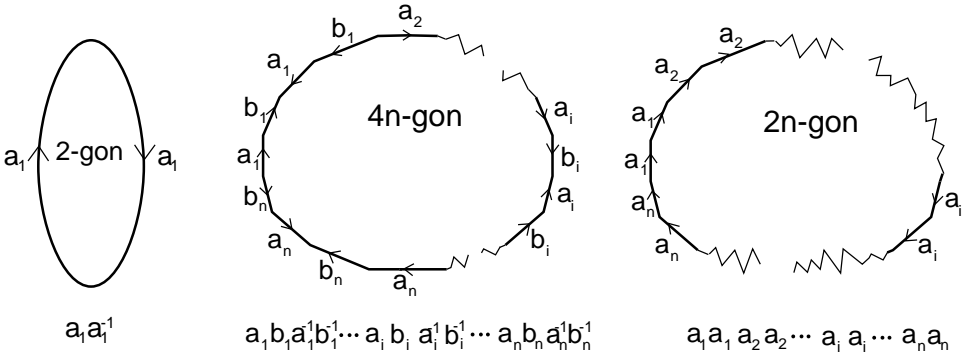
Possibility (b)



Case 2. Distinct triangles adjacent to the edge  $x$  (continued)



**Definition.** A planar polygon giving a 2-manifold is said to be in **canonical form** if it has edge identifications one of the following:



**Definition.** A triangulation of a 2-manifold obtained from a minimal triangulation of a canonical form is called a **canonical triangulation** of the 2-manifold. Such a triangulation has only one vertex.

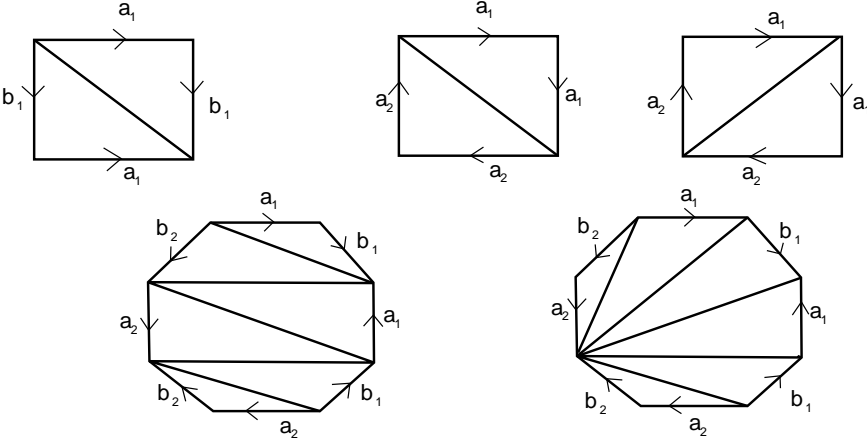


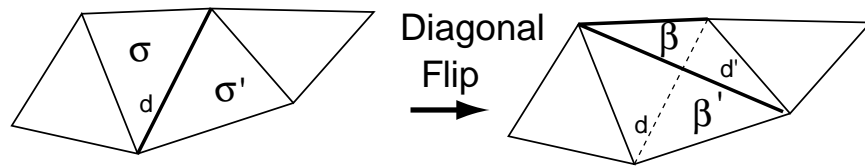
FIGURE: Examples of Canonical Triangulations

**Proposition.** *There is an algorithm modifying any one-vertex triangulation of a 2-manifold to a canonical triangulation.*

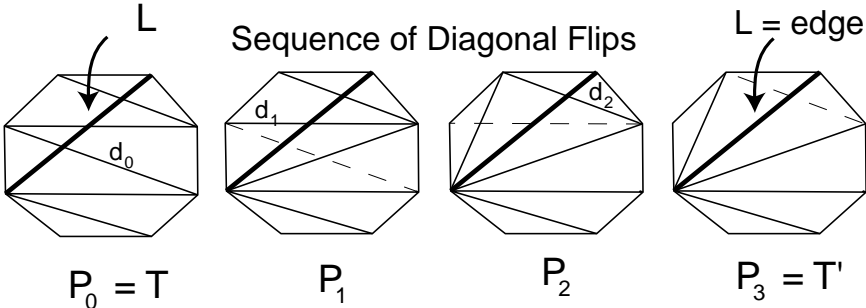
**PROOF:**

**Step 1.** Any one-vertex triangulation of a 2-manifold is obtained from a minimal triangulation of some regular  $p$ -gon with its boundary edges identified in pairs. Uses PL-Jordan-Schoenflies Theorem (Jordan, 1887; Veblen, 1905; Schoenflies, 1907).

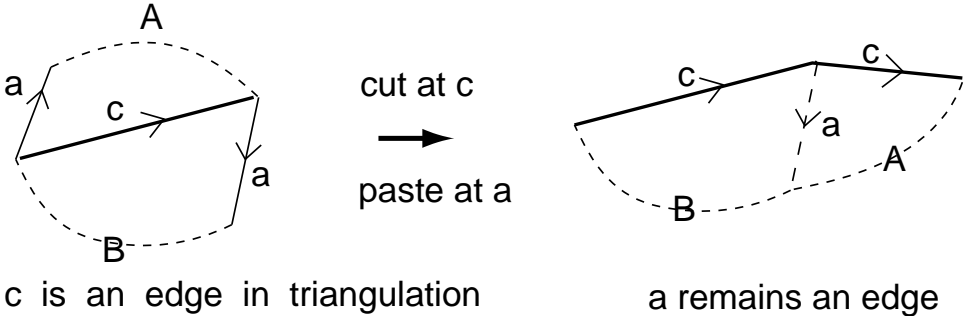
**Definition.** Diagonal Flip or a  $2 \leftrightarrow 2$  Pachner move



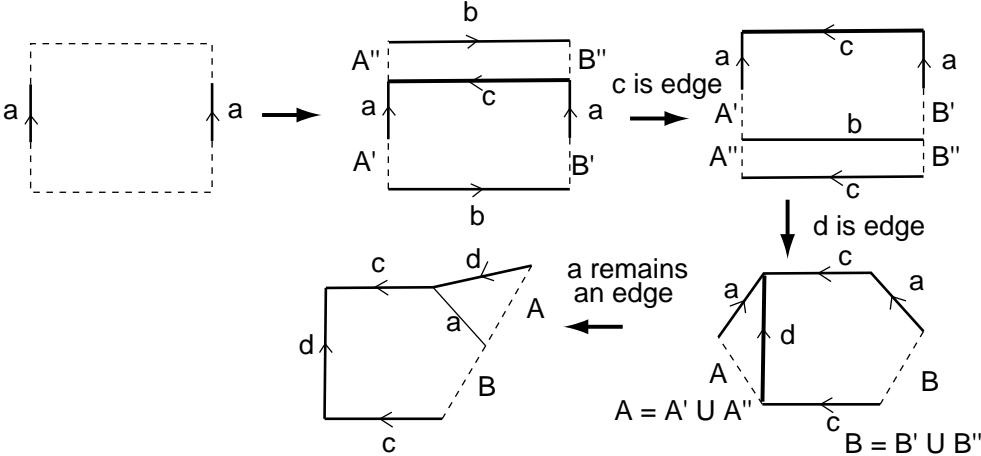
**Lemma.** Suppose  $\mathcal{T}$  is a triangulation of a regular  $p$ -gon having all its vertices in the boundary and  $L$  is a line segment between vertices of  $\mathcal{T}$ . Then there is a triangulation  $\mathcal{T}'$  of the regular  $p$ -gon having  $L$  as an edge and a sequence of triangulations  $\mathcal{T} = \mathcal{P}_0, \mathcal{P}_1 \dots, \mathcal{P}_k = \mathcal{T}'$ , where  $\mathcal{P}_i$  is obtained from  $\mathcal{P}_{i-1}$  by a diagonal flip  $1 \leq i \leq k$ .



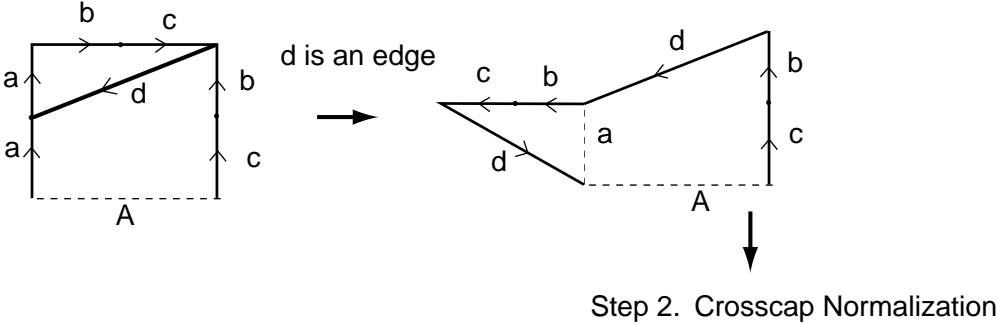
**Step 2.** Cross cap Normalization. Identified pair of edges oriented in same direction can be replaced by adjacent such pair.



**Step 3.** Handle Normalization. Crossed pairs may be replaced by handles.



**Step 4.** Metamorphosis of handle to Crosscaps ( $1 + 1 = 3$ ). **One** handle plus **one** crosscap = **three** crosscaps



**CONCLUSION:** The LIST is complete (every 2-manifold is on the list).

**Proposition.** *The “LIST” is without repeats.*

NOTE: This uses topological invariance of fundamental group.

**Lemma.** *We have:*

1. *The fundamental group of  $S^2$  is trivial.*
2. *A presentation for the fundamental group of the orientable surface,  $F_n^+$  of genus  $n, n \geq 1$  is:*

$$\pi_1(F_n^+) = \{a_1, b_1, a_2, b_2, \dots, a_n, b_n \mid \prod_{i=1}^n [a_i, b_i]\}.$$

3. *A presentation for the fundamental group of the nonorientable surface,  $F_n^-$  of genus  $n, n \geq 1$  is:*

$$\pi_1(F_n^-) = \{a_1, a_2, \dots, a_n \mid \prod_{i=1}^n a_i^2\}.$$

NOTE: This uses a simple version of the Seifert-Van Kampen Theorem.

**Euler Characteristic**,  $\chi = f - e + v$ , where  $f$  is number of triangles,  $e$  is number of edges and  $v$  is number of vertices.

$$\chi = f - e + v = -f/2 + v \Rightarrow f = 2(v - \chi)$$

**Proposition.** *Euler Characteristic is a topological invariant for 2-manifolds.*

**Proof:**

- Closing-the-book preserves Euler characteristic
- Diagonal Flips, Canonical Modifications of a minimal triangulation does not change Euler characteristic.
- Triangulations  $\mathcal{T}$  and  $\mathcal{T}'$  of  $F$ , are modified to a minimal triangulation of a canonical form without changing Euler characteristic.
- Distinct canonical  $\Rightarrow$  distinct 2-manifolds
- Minimal triangulations of a fixed canonical form have same Euler characteristic.

**Theorem. (HOMEOMORPHISM PROBLEM)**

Given 2-manifolds  $F$  and  $F'$ , they are homeomorphic if and only if they have the same orientability class and the same Euler characteristic.

Euler Char.	Orientable	Non-orientable
$\chi = 2$	$S^2$	
$\chi = 1$		$\mathbb{R}P^2$
$\chi = 0$	$T^2 = S^1 \times S^1$	$\mathbb{R}P^2 \# \mathbb{R}P^2$
$\chi = -1$		$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$
$\chi = -2$	$T^2 \# T^2$	$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$\chi = -2n$	$\underbrace{T^2 \# \dots \# T^2}_{n\text{-terms}}$	$\underbrace{\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2}_{2n\text{-terms}}$
$\vdots$	$\vdots$	$\vdots$

## Why this does NOT work for 3–manifolds

- The Euler characteristic of a closed 3–manifold is 0.
  - A closed 3–manifold has a one-vertex triangulation; in fact, a one-vertex-triangulation with an arbitrarily large number of tetrahedra.
- There are no candidates analogous to a canonical form for 2–manifolds.
  - A closed 3–manifold can be obtained from a triangulated 3–cell in  $\mathbb{R}^3$  with its boundary faces identified in pairs.

**Theorem (Markov, 1958).** *The Homeomorphism Problem is **unsolvable** for triangulated  $n$ -manifolds,  $n \geq 4$ .*

1. For any finite 2-complex  $K$ , there is a 4-manifold  $M(K)$  such that  $\pi_1(M(K)) \approx \pi_1(K)$ .
2. Given a presentation  $\mathcal{P}$  of a group,  $Group(\mathcal{P})$ , there is a canonical 2-complex, the surface complex,  $S(\mathcal{P})$ , that can be constructed with  $\pi_1(S(\mathcal{P})) = Group(\mathcal{P})$ ; thus a 4-manifold  $M(S(\mathcal{P}))$  with  $\pi_1(M(S(\mathcal{P}))) \approx Group(\mathcal{P})$ .
3. Given presentation  $\mathcal{P}$  and  $\mathcal{P}'$ , there are computable non negative integers  $p, p'$  so that if  $Group(\mathcal{P}) \approx Group(\mathcal{P}')$ , then the presentation  $\mathcal{P} * p$  can be converted to the group presentation  $\mathcal{P}' * p'$  by special Tietze transformations.

4. If the presentation  $\mathcal{P} * p$  can be converted to the group presentation  $\mathcal{P}' * p'$  by special Tietze transformations, then  $M(S(\mathcal{P} * p))$  is homeomorphic to  $M(S(\mathcal{P}' * p'))$ .

Hence,

$$\text{Group}(\mathcal{P}) \approx \text{Group}(\mathcal{P}') \Leftrightarrow$$

$$\text{Group}(\mathcal{P} * p) \approx \text{Group}(\mathcal{P}' * p') \Leftrightarrow$$

$$M(S(\mathcal{P} * p)) \text{ is homeomorphic to } M(S(\mathcal{P}' * p'))$$

## DETAILS:

- Suppose we can decide the Homeomorphism Problem for triangulated 4–manifolds.
- Let  $\mathcal{P}$  and  $\mathcal{P}'$  be any two finite group presentations.
- Compute  $p$  and  $p'$  where  $\mathcal{P} * p$  can be converted to  $\mathcal{P}' * p'$  by special Tietze transformations if  $Group(\mathcal{P}) \approx Group(\mathcal{P}')$ .
- Construct  $M(S(\mathcal{P} * p))$  and  $M(S(\mathcal{P}' * p'))$ .

If  $M(S(\mathcal{P} * p))$  is homeomorphic to  $M(S(\mathcal{P}' * p'))$ , then  $Group(\mathcal{P}) \approx Group(\mathcal{P}')$ .

If  $M(S(\mathcal{P} * p))$  is NOT homeomorphic to  $M(S(\mathcal{P}' * p'))$ , then  $Group(\mathcal{P}) \not\approx Group(\mathcal{P}')$ .

BUT this gives a solution to the Isomorphism Problem for finitely presented group. A Contradiction.