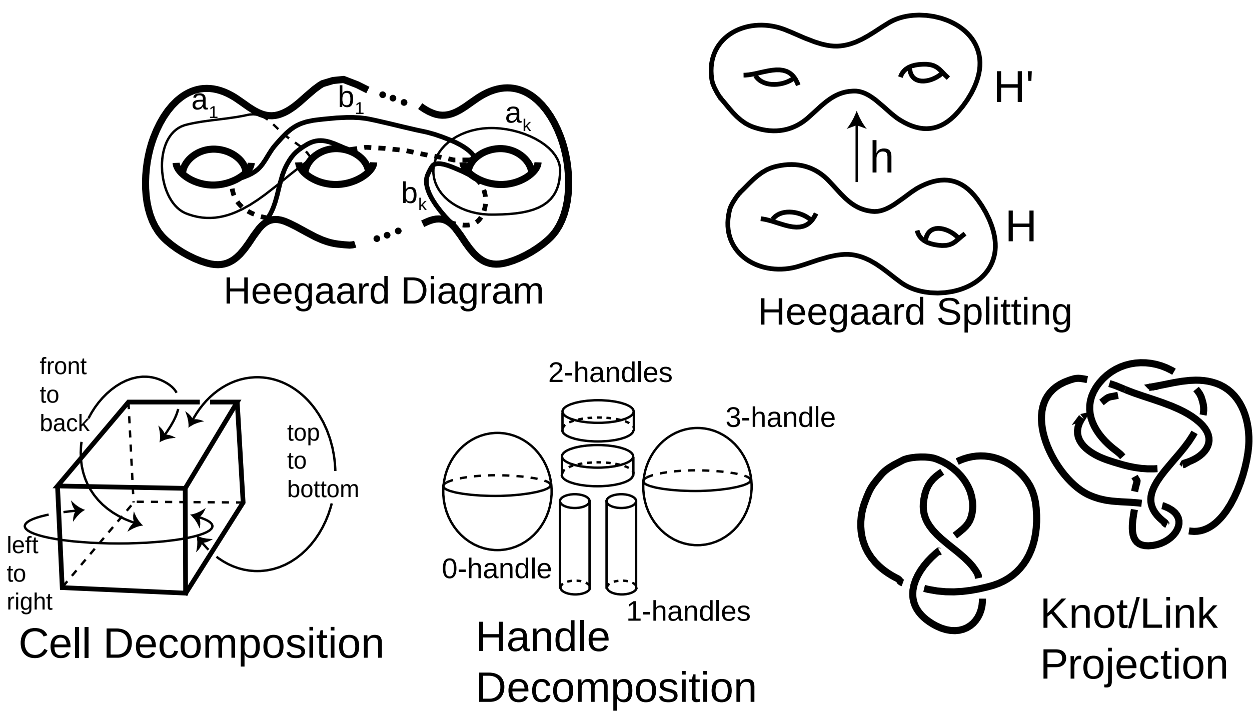
IIA-B. PRESENTATIONS OF 3–MANIFOLDS

Abstract: These two lectures will explore the various presentations of 3–manifolds via triangulations, cell-decompositions, handle-decompositions, Heegaard splitings, Heegaard diagrams, and knot and link projections. Algorithms will be given that transform each of these presentations into a triangulation.

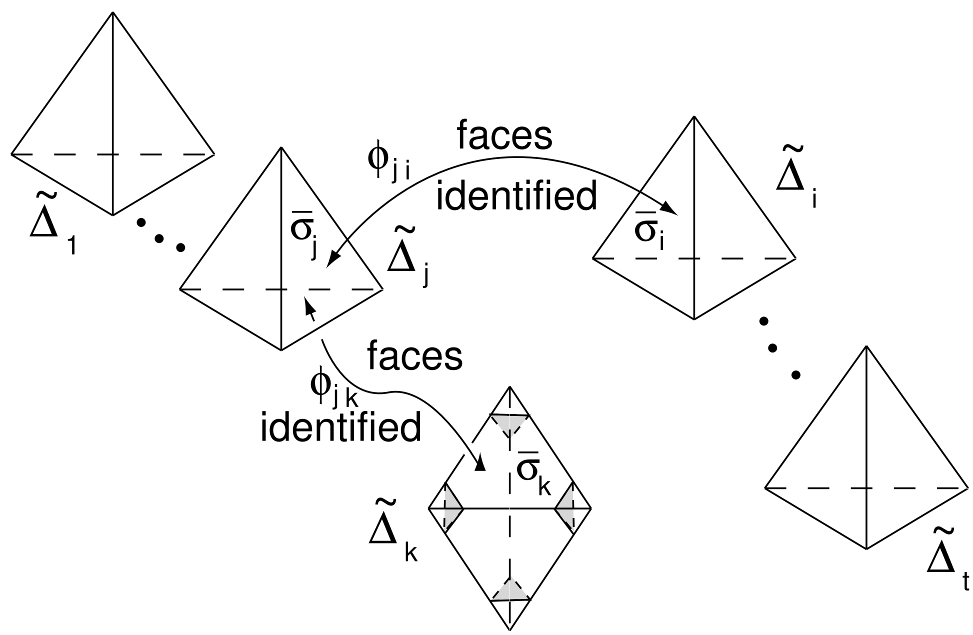
PROBLEM: To convert a presentation of a 3–manifold via a Heegaard diagram, Heegaard splitting, cell-decomposition, handle-decomposition, or knot or link projection to a triangulation.



Definition. A **triangulation** $\mathcal{T}$ of a 3–manifold M is a pairwise disjoint collection of tetrahedra, $\mathbf{\Delta}$, along with a family of face identifications, Φ, so that the associated identification space $|\mathcal{T}| = \mathbf{\Delta}/\Phi = M$.

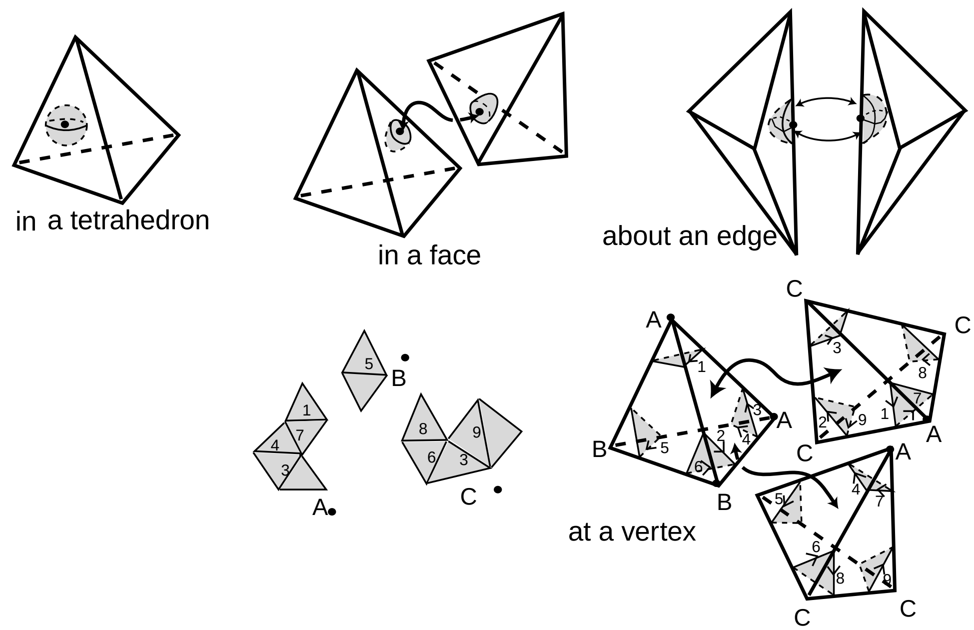
$\mathbf{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$, pairwise disjoint family of tetrahedra.

Φ, is a family of affine isomorphisms pairing distinct faces of the tetrahedra in $\mathbf{\Delta}$ so that $\phi \in \Phi$, then ϕ is a simplicial homeomorphism from a face of some $\tilde{\Delta}_i$ to a face of some $\tilde{\Delta}_j$, possibly $i = j$.



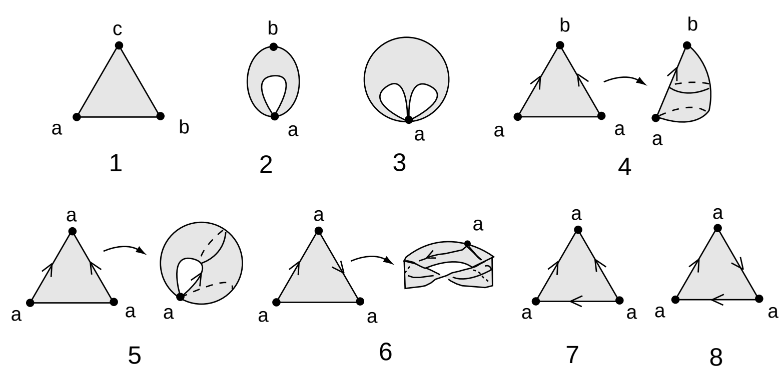
Definition. A *tetrahedron, face, edge*, or *vertex* in this cell decomposition is, respectively, the image of a tetrahedron, face, edge, or vertex from the collection $\mathbf{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$.

3-dimensional neighborhoods



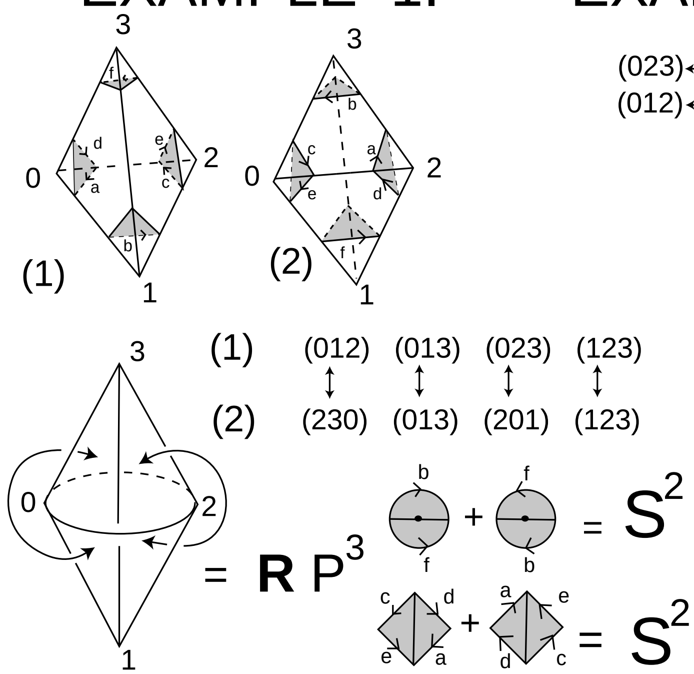
Discuss: Probability of getting a 3–manifold.

Note: Identification map is an embedding on the interior of simplicies; edges can go to simple closed curves, faces to cones, Möbius bands, etc.

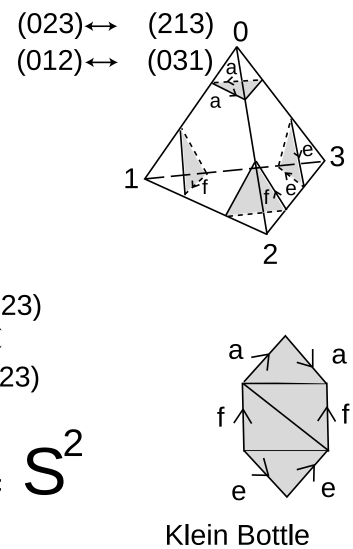


EXAMPLES:

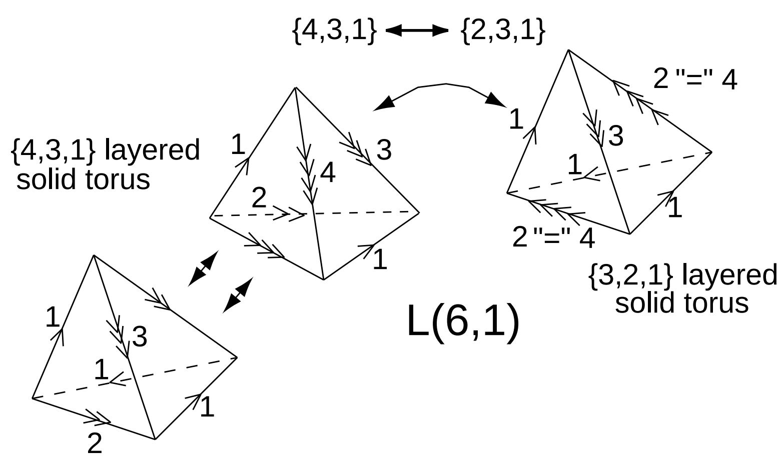
EXAMPLE 1.



EXAMPLE 2.



EXAMPLE 3.



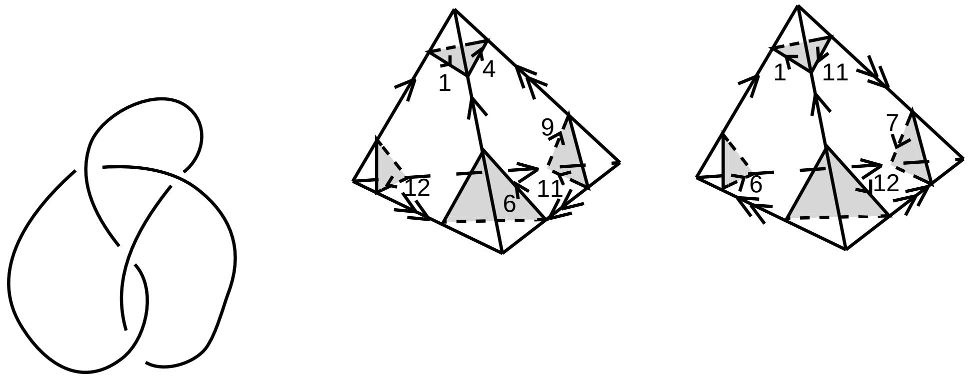
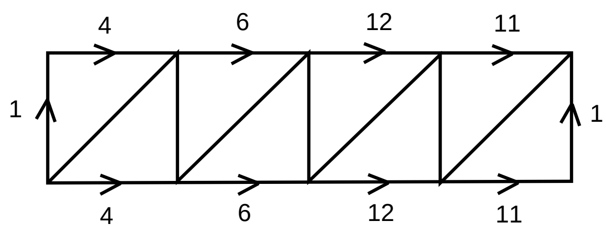


FIGURE EIGHT KNOT



VERTEX-LINKING TORUS

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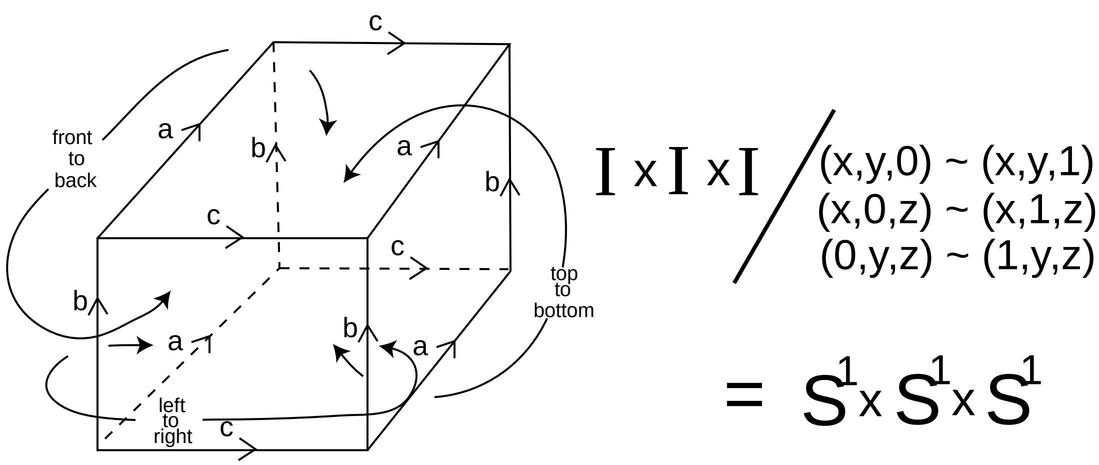
Definition. If $|\mathcal{T}| = \mathbf{\Delta}/\Phi$ is a manifold except at vertices, then $|\mathcal{T}| - \mathcal{T}^{(0)} = \mathbf{\Delta}/\Phi-$vertices $=\overset{\circ}{X}$ is the interior of a compact 3–manifold X and we call $\mathcal{T}$ an **ideal triangulation** of $\overset{\circ}{X}$. The vertices are **ideal vertices** and the **index** of an ideal vertex is the genus of the vertex-linking surface.

Lemma. *The second derived subdivision of a triangulation is a rectilinear (Piecewise Linear) triangulation; i.e., embeds as a subcomplex of* $\mathbb{R}^n$.

CELL-DECOMPOSITIONS:

EXAMPLE:



Proposition. *A cell-decomposition can be subdivided to a triangulation without adding vertices.*

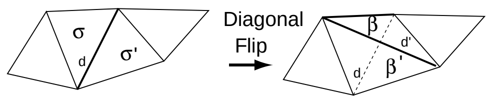
Proof:

STEP 1. Triangulate each cell - before identification - without adding vertices.

Lemma. *A compact convex linear cell can be triangulated without adding vertices.*

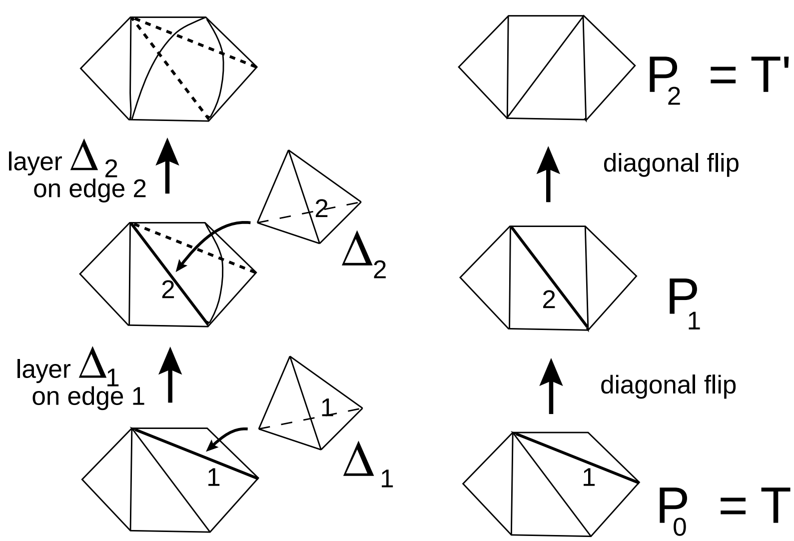
STEP 2.

Definition. Diagonal Flip or a $2 \leftrightarrow 2$ Pachner move



Lemma. *Suppose* $\mathcal{T}$ *and* $\mathcal{T}'$ *are triangulations of a regular* p*–gon having all its vertices in the boundary, then there is a sequence of triangulations* $\mathcal{T} = \mathcal{P}_0, \mathcal{P}_1 \dots, \mathcal{P}_k = \mathcal{T}'$*, where* $\mathcal{P}_i$ *is obtained from* $\mathcal{P}_{i-1}$ *by a diagonal flip* $1 \le i \le k$*.*

PLEATING COMBINATORIAL STRUCTURES
Diagonal Flips and $2 \leftrightarrow 2$ Pachner Moves



HEEGAARD SPLITTINGS:

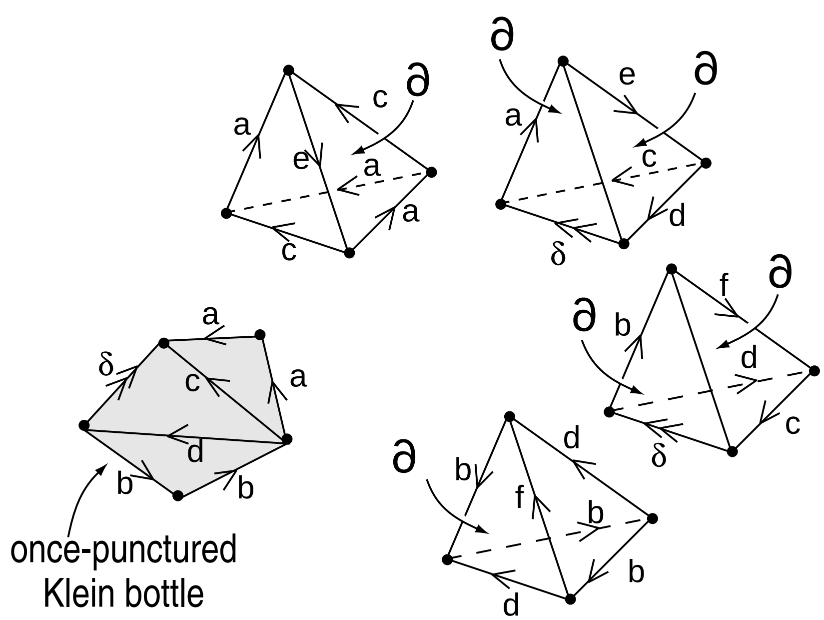
Definition. Handlebody, genus, handles, complete set of curves, complete set of handles

Theorem. *Every closed, orientable* 3*−manifold* M *can be written as the union* $M = H \cup H'$*, where* H *and* H' *are handlebodies and* $H \cap H' = \partial H = \partial H' = S$*.*

Proof:
Theorem (Moise,1952;Bing,1959). 3*−manifolds can be triangulated. Furthermore, any two triangulations of the same* 3*−manifold have isomorphic subdivisions.*

Definition. $(M; H, H')$ or $(M; S)$ Heegaard Splitting of M, genus of splitting, genus of M, S Heegaard surface

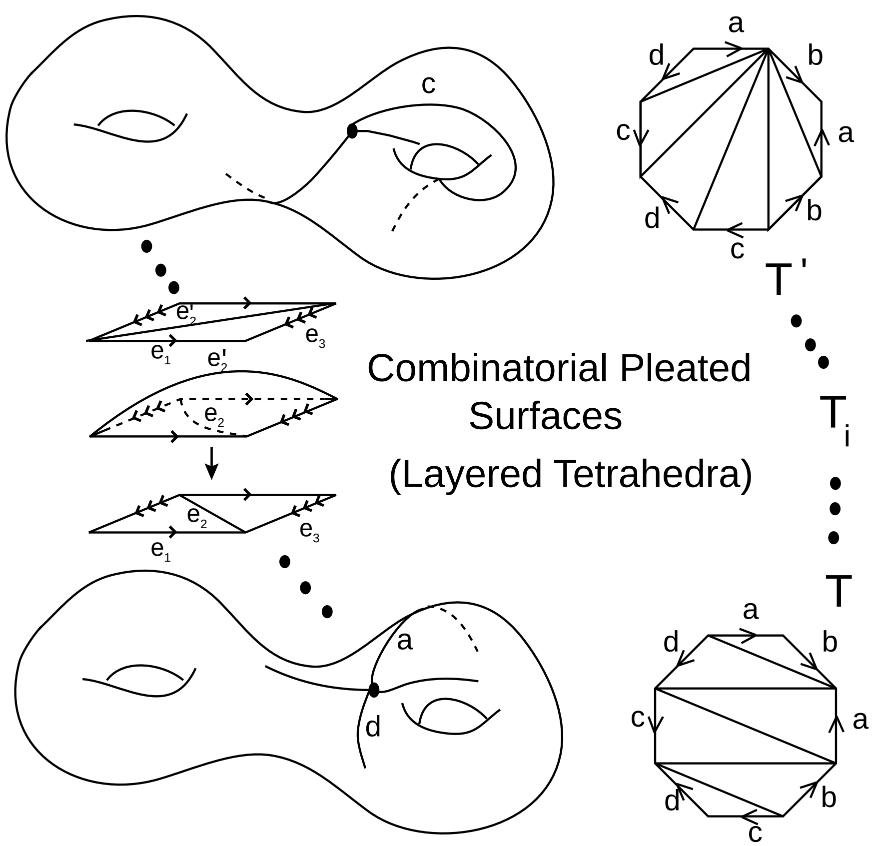
Proposition. *A handlebody has a one-vertex triangulation. Furthermore, a minimal triangulation is a one-vertex triangulation and the minimal number of tetrahedra necessary to triangulate a genus g handlebody is $3g-2$.*



genus 2 handlebody - layered triangulation

Theorem. *Given a Heegaard Splitting of the 3–manifold M, there is an algorithm to construct a one-vertex triangulation of M.*

Proof.



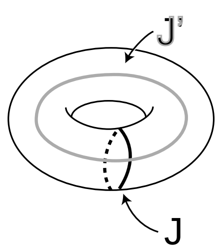
Corollary. *Every closed 3–manifold admits a one-vertex triangulation. In fact, for any positive integer C a 3–manifold has a one-vertex triangulation with $\geq C$ tetrahedra.*

HEEGAARD DIAGRAM:

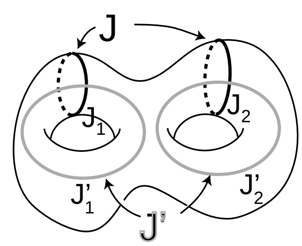
Definition. complete set of curves on a surface; Heegaard Diagram $(S; J, J')$

EXAMPLES:

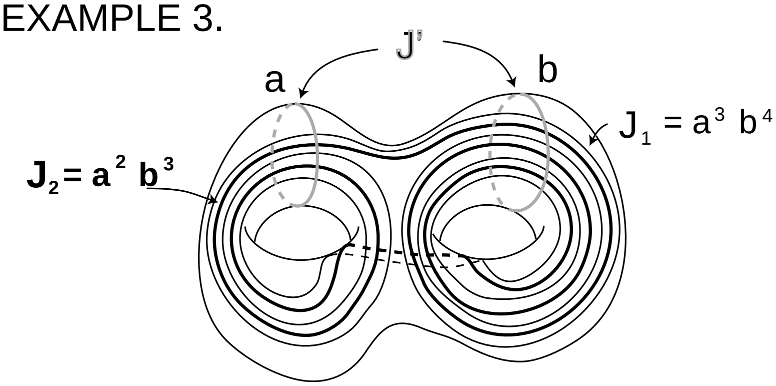
EXAMPLE 1.



EXAMPLE 2.



EXAMPLE 3.



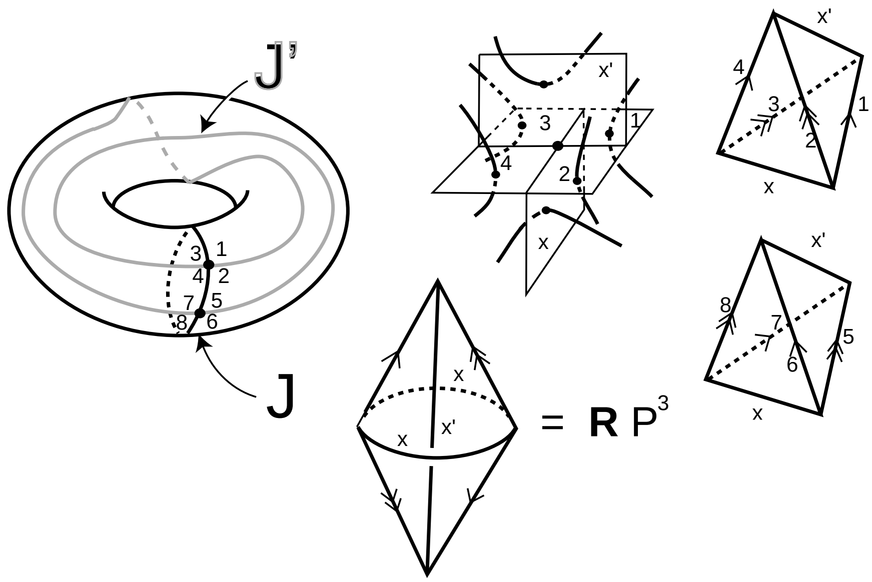
OBSERVE:

1. A Heegaard Diagram determines a unique (closed, orientable) 3–manifold $M = M(S; J, J')$ and a Heegaard splitting of M with Heegaard surface S.

   Definition. reducible Heegaard Splitting, full Heegaard diagram

2. If the Heegaard diagram $(S; J, J')$ is full, then it determines a cell-decomposition of the 3–manifold $M(S; J, J')$ called the **associated cell-decomposition**.

Proposition. *Suppose* $(S; J, J')$ *is a full Heegaard diagram for the* 3*–manifold* $M = M(S; J, J')$ *and* $\mathcal{C}$ *is the associated cell-decomposition. Then the cell-decomposition dual to* $\mathcal{C}$ *is a two-vertex triangulation of* M*.*



OBSERVE: The Heegaard surface meets each tetrahedron in a single quadrilateral.

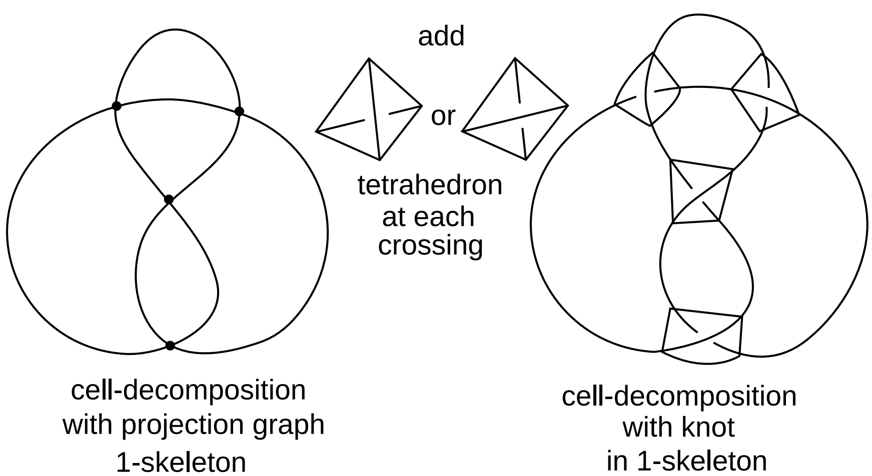
Corollary. *Given a (full) Heegaard diagram of the* 3*–manifold* M*, there is an algorithm to present* M *with a two-vertex triangulation.*

KNOT AND LINK PROJECTIONS:

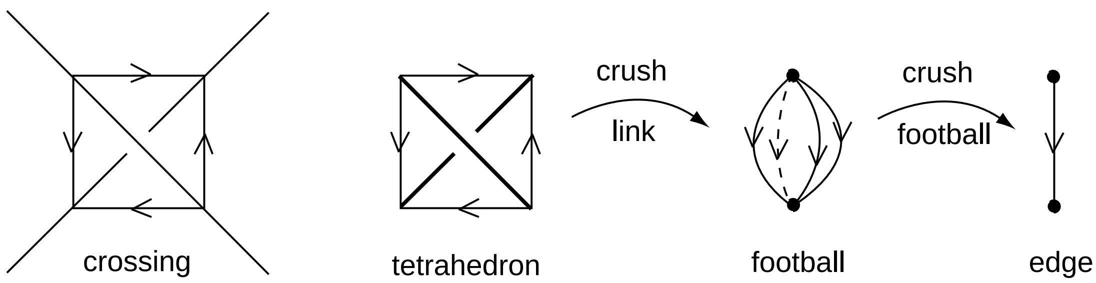
Definition. knot (link) complement, knot (link) exterior

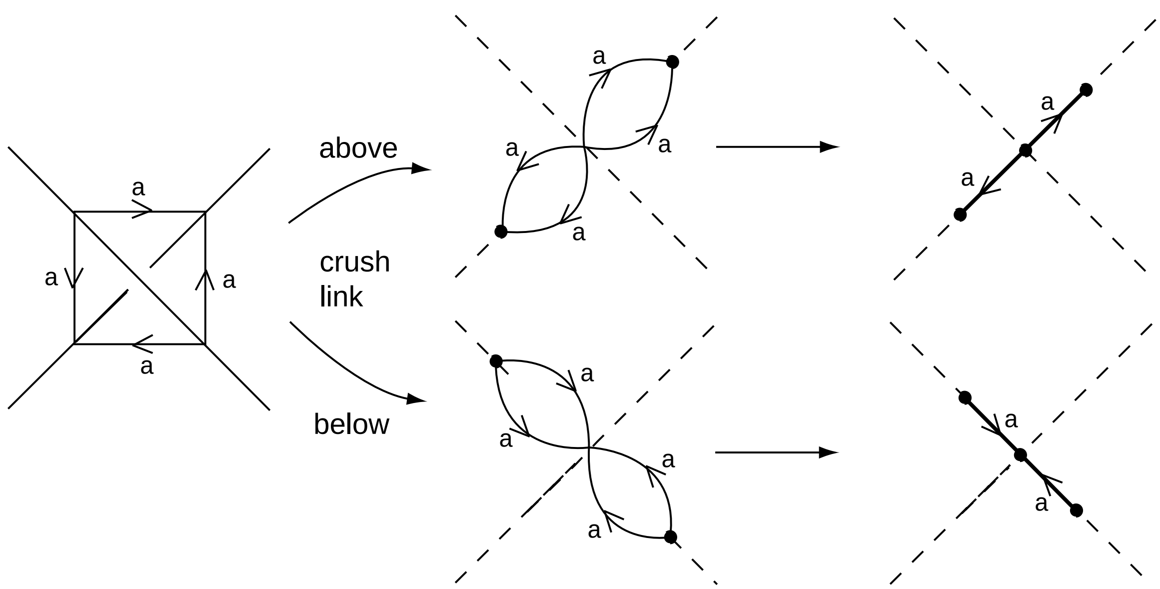
Theorem. *Given a knot (or link) projection, there is an algorithm to construct an ideal triangulation of the knot (or link) complement.*

STEP 1. Cell decomposition of S^3 with knot (link) in 1–skeleton.

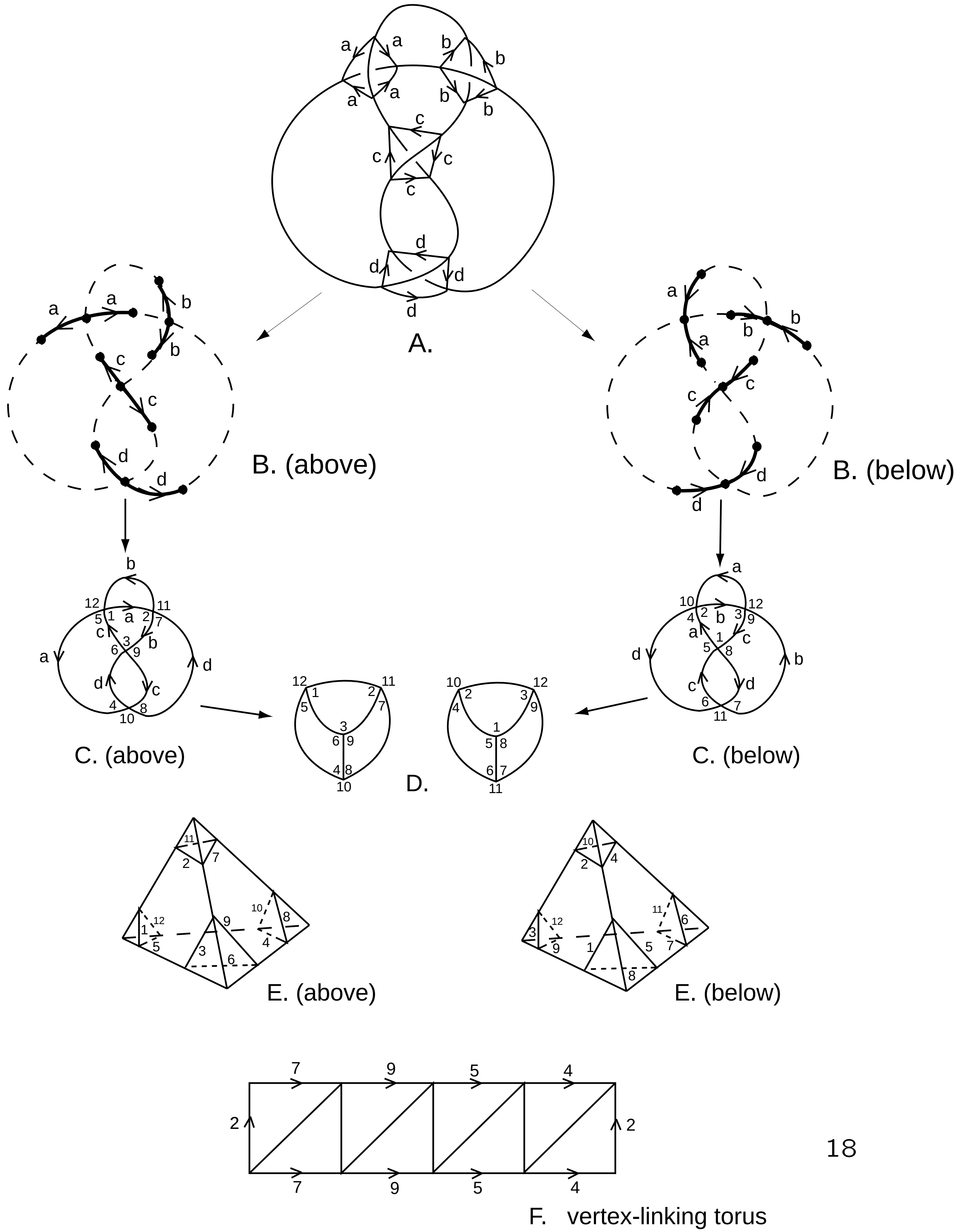


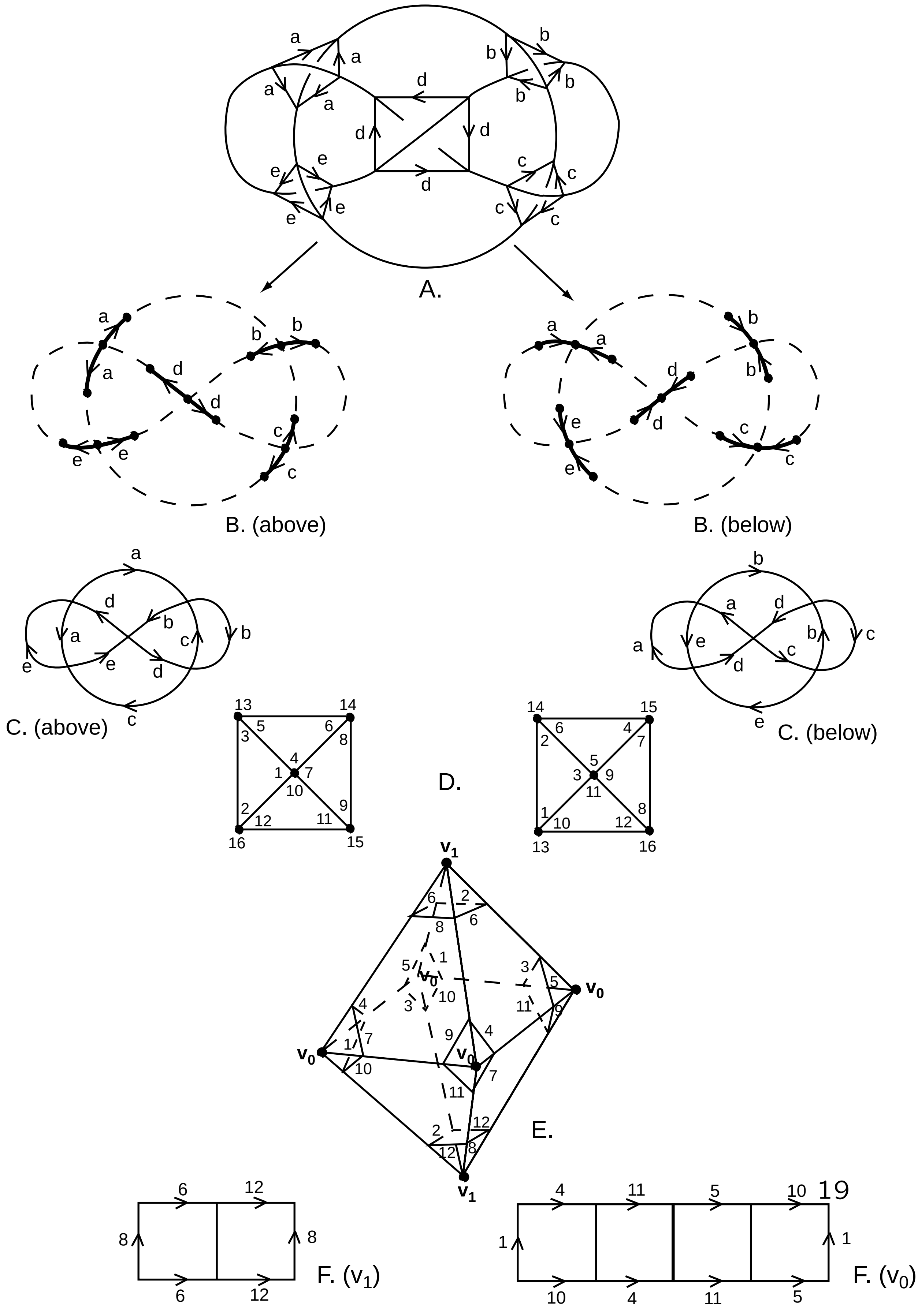
Step 2. Crush knot (link).



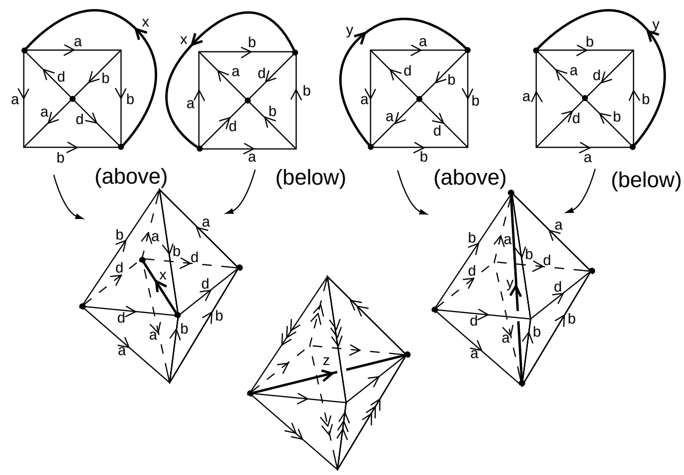


Get *ideal* cell-decomposition of knot (link) complement. An ideal vertex for each component.

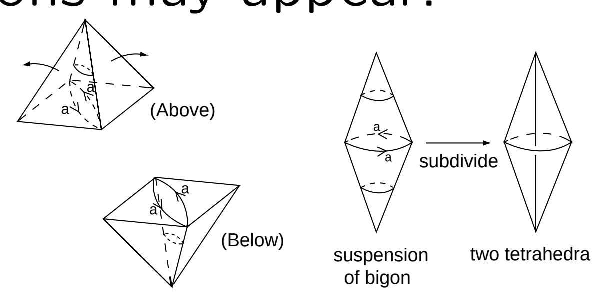

A.
B. (above)
B. (below)
C. (above)
C. (below)
D.
E. (above)
E. (below)
F. vertex-linking torus


A.
B. (above)
B. (below)
C. (above)
C. (below)
D.
E.
F. (v_1)
F. (v_0)

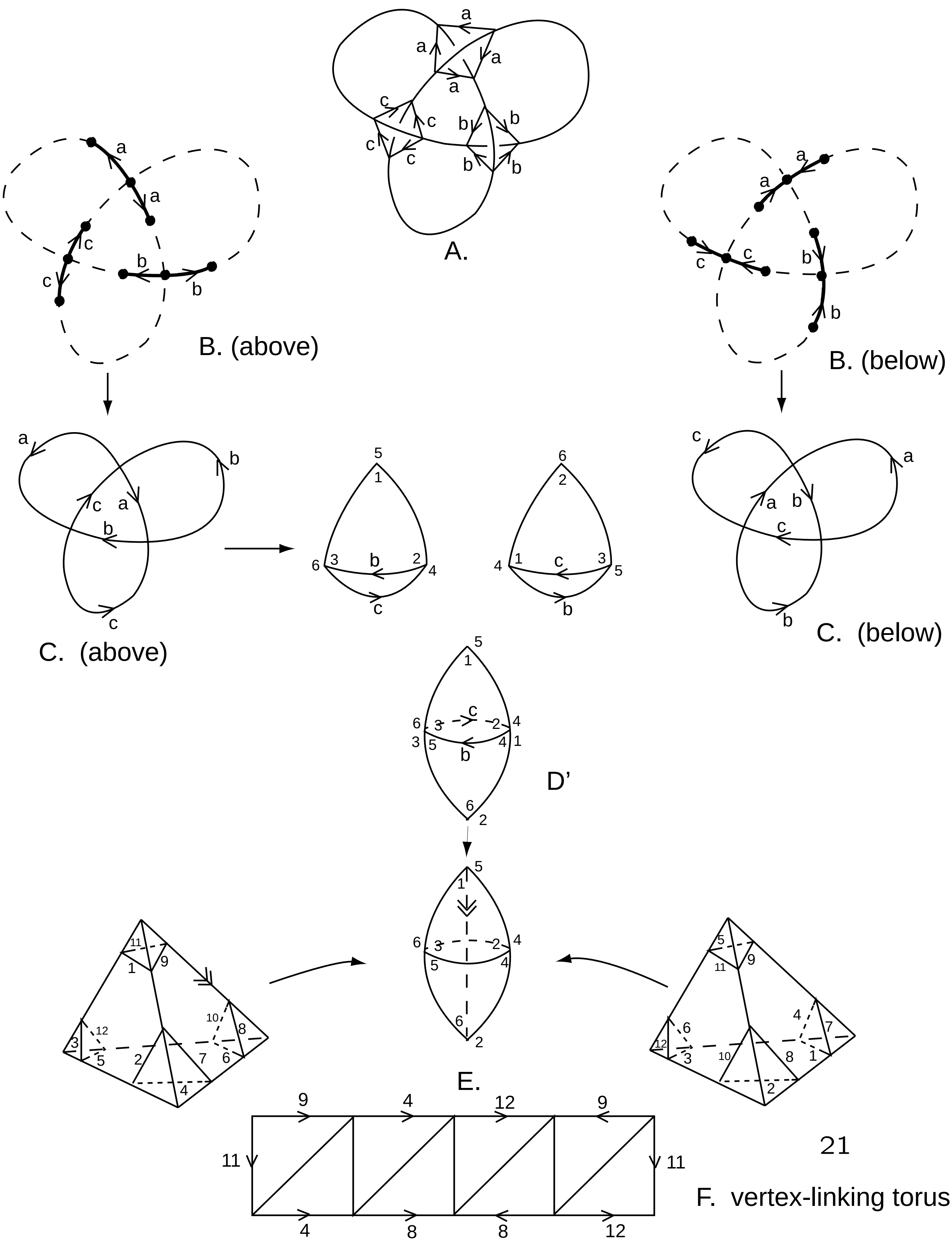
RECALL: Given a cell-decomposition (ideal cell-decomposition) of a 3–manifold M (of the interior, $\overset{\circ}{M}$, of a 3–manifold M), there is an algorithm to construct a triangulation (ideal-triangulation) of M (of $\overset{\circ}{M}$), without adding vertices.



NOTE: Bigons may appear.

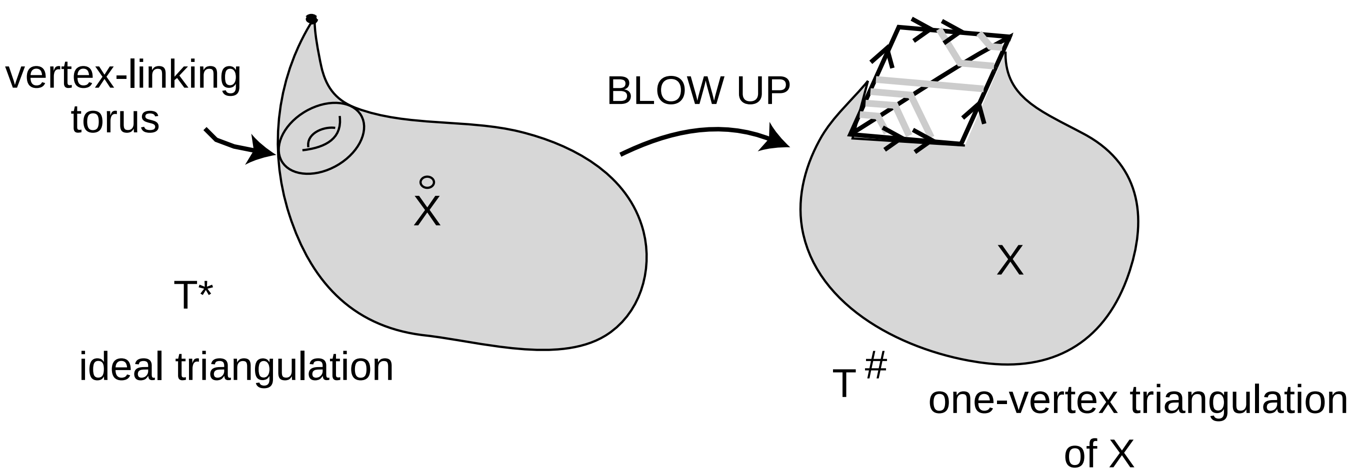


Thus, given a knot (or link) projection, there is an algorithm to construct an ideal triangulation of the knot (or link) complement.

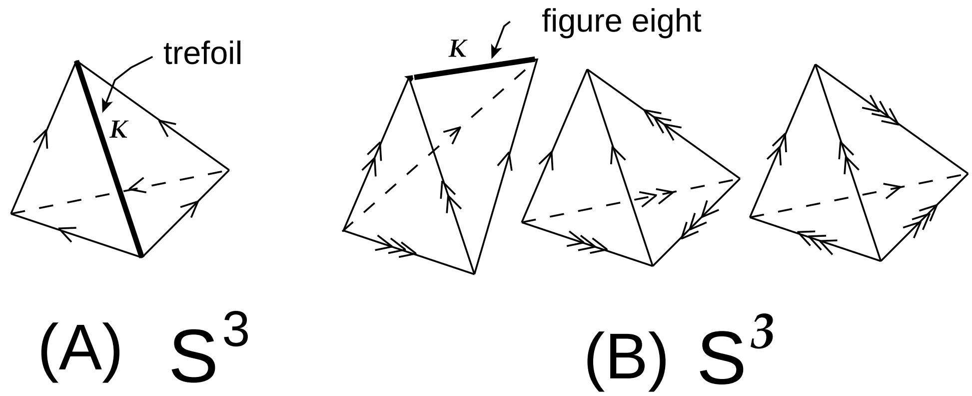

A.
B. (above)
B. (below)
C. (above)
C. (below)
D’
E.
F. vertex-linking torus

Brief discussion of slopes, Dehn-fillings, exceptional slopes, edge-slopes, crushing, blow-ups, meridional/longitude as edge-slopes, etc.

Proposition. *Given an ideal triangulation of a knot complement, there is a blow-up of the ideal triangulation to a one-vertex triangulation of the knot exterior (a compact* 3*−manifold with boundary).*



Proposition. *Given any knot in S^3, there is a one-vertex triangulation of S^3 having the knot as a single edge.*



Brief discussion of layered triangulations of solid tori, canonical triangulations for Dehn-fillings, etc.