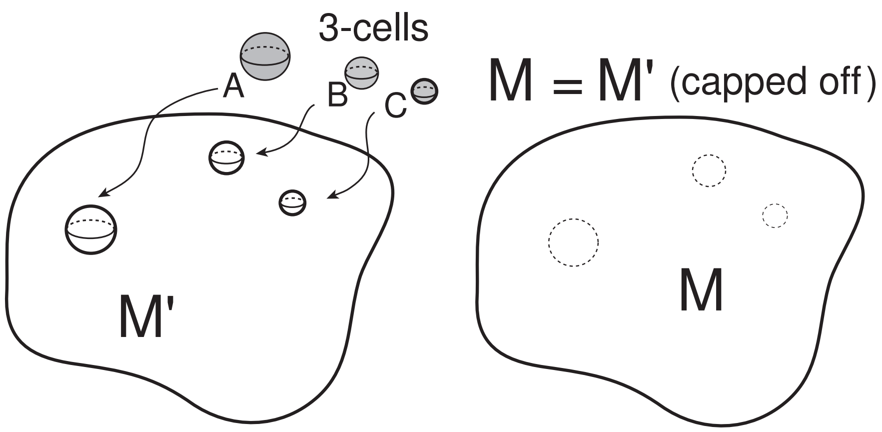
IV A. Prime Decomposition of 3–manifolds

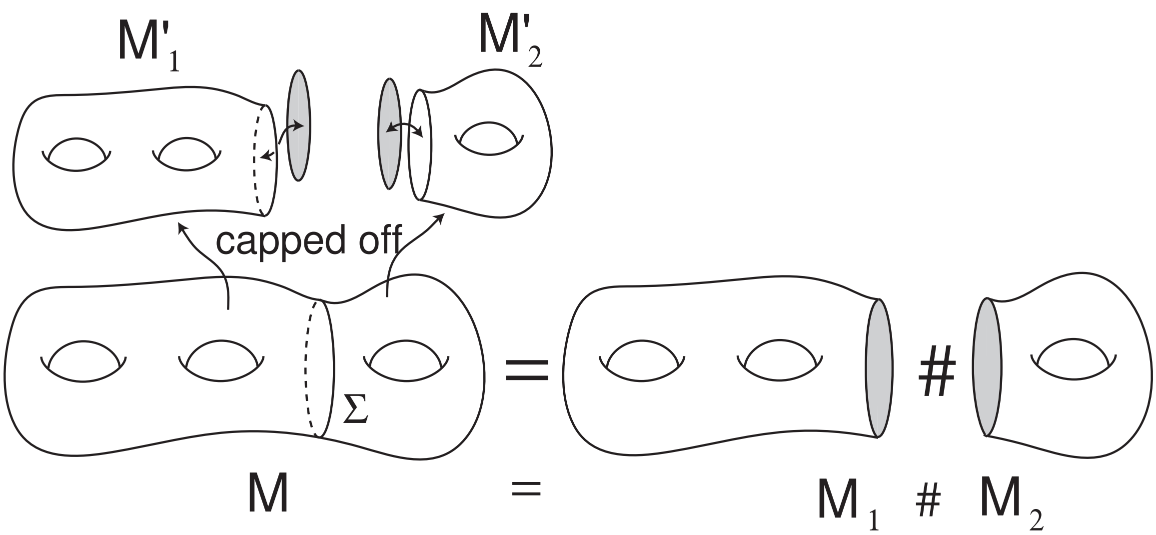
Abstract: This lecture will provide the existence and uniqueness theorems of H.Kneser and J.Milnor for the prime decomposition of 3–manifolds.

Definition. connected sum, capping-off, prime, irreducible, punctured 3–sphere, independent 2–spheres

Capping-off 2–sphere boundary components:



Connected-Sum

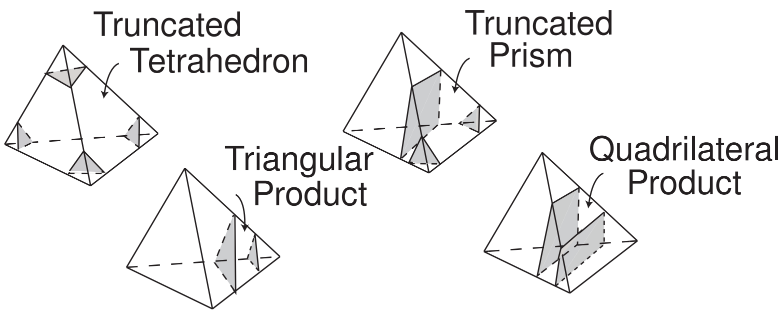


Proposition (Kneser-Haken Finiteness). *Suppose M is a closed 3–manifold and $\mathcal{T}$ is a triangulation of M having t tetrahedra. Furthermore, suppose $F_1, \ldots, F_K$ is a pairwise disjoint collection of distinct normal surfaces none of which are vertex-linking or edge-linking. Then $K \leq 2t$.*

Proof:(Outline)

• At each vertex, remove an open vertex-linking 3–cell, getting the manifold M^* with v (the number of vertices) 2–spheres in its boundary.

• Split M^* along $F_1, \ldots, F_K$; result has "nice" cell-decomposition.



Proof:(Outline Continued)

Definition. "good" and "bad" components; "good" and "bad" remnants; "good" and "bad" normal triangles and quads; "bad disks"

• $2K = g + b$, g is number of "good" remnants; b is number of "bad" remants

• the number of "bad" normal triangles, $s \leq 4t$ and number of "bad" normal quads $q \leq 2t$

• the number of "good" remnants $g \leq t + 1$

• There are at least two "bad" disks in each remnant; so, $b \leq (s + q)/2 \leq 3t$

• $2K \leq g + b \leq t + 1 + 3t = 4t + 1$; so, $K \leq 2t$.

Discuss Examples.

QUESTION: Can we improve this number; e.g., $t/2$ should suffice?

Lemma. *Suppose M is a 3–manifold and $F_1, \ldots, F_K$ is a pairwise disjoint collection of independent 2–spheres. Then for any triangulation $\mathcal{T}$ of M, there is a pairwise disjoint collection of K independent, normal 2–spheres.*

Theorem (Existence of Prime Decomposition (Kneser, 1927))**.** *A compact 3–manifold $M \neq S^3$ can be written as a connected sum $M = M_1 \# \ldots \# M_n$, where each M_i is prime ($M_i \neq S^3$).*

Theorem (Uniqueness of Prime Decomposition (Milnor, 1962)). *If* $M = M_1 \# \ldots \# M_n$ *and* $M = M'_1 \# \ldots \# M'_{n'}$ *are prime decompositions of the 3–manifold* M*, then* $n' = n$ *and after possibly reordering,* M'_i *is homeomorphic to* $M_i, 1 \leq i \leq n$*.*

NOTE: A prime decomposition of $M \neq S^3$ can be written $M = M_1 \# \ldots \# M_\ell \# m(S^2 \times S^1)$, where each $M_i \neq S^3, 1 \leq i \leq \ell$ is irreducible.

RESTATEMENT: If $M = M_1 \# \ldots \# M_\ell \# m(S^2 \times S^1)$ and $M = M'_1 \# \ldots \# M'_{\ell'} \# m'(S^2 \times S^1)$ are prime decompositions of M, where each $M_i \neq S^3, 1 \leq i \leq \ell$, and each $M'_i \neq S^3, 1 \leq i \leq \ell'$, is irreducible, then $\ell' = \ell, m' = m$ and after possibly reordering, M'_i is homeomorphic to $M_i, 1 \leq i \leq \ell$

Proof: (Outline)

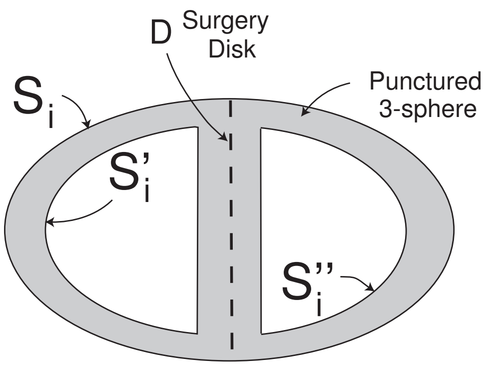
• There is a pairwise disjoint collection $\mathcal{S}$ of 2–spheres embedded in M so that the closure of a component of $M \backslash \mathcal{S}$ is a punctured $M_i, 1 \leq i \leq \ell$ or a punctured 3–sphere.

• If $\mathcal{S}'$ is a pairwise disjoint collection of 2–spheres embedded in M and $\mathcal{S}' \supseteq \mathcal{S}$, then the closure of a component of $M \backslash \mathcal{S}'$ is a punctured $M_i, 1 \leq i \leq \ell$ or a punctured 3–sphere.

• There is a collection $\mathcal{R}$ of such 2–spheres for the decomposition $M = M_1' \# \ldots \# M_{\ell'}' \# m'(S^2 \times S^1)$ so that the closure of a component of $M \backslash \mathcal{R}$ is a punctured $M_j' (1 \leq j \leq \ell')$ or a punctured 3–sphere.

Proof: (Outline continued)

• If S_i' and S_i'' are obtained by surgery on S_i, then the closure of each component of the complement of the collection $(\mathcal{S} \setminus S_i) \cup \{S_i', S_i''\}$ is a punctured $M_i, 1 \le i \le \ell$, or a punctured 3–sphere.



• There are two collections of pairwise disjoint 2–spheres embedded in M, say $\mathcal{S}'$ and $\mathcal{R}'$, so that $\mathcal{S}' \cap \mathcal{R}' = \emptyset$ and the closure of a component of $M \setminus \mathcal{S}'$ is a punctured $M_i, 1 \le i \le \ell$ or a punctured 3–sphere; whereas, the closure of a component of $M \setminus \mathcal{R}'$ is a punctured $M_i', 1 \le i \le \ell'$ or a punctured 3–sphere.

Proof: (Outline continued)

• The closure of a component of $\mathcal{S}' \cup \mathcal{R}'$ is a punctured $M_i, 1 \leq i \leq \ell$ or a punctured 3–sphere but is also a punctured $M_i', 1 \leq i \leq \ell'$ or a punctured 3–sphere.

• Thus for some reordering $M_i' = M_i, 1 \leq i \leq \ell$, we have $M = M_1 \# \ldots \# M_\ell \# m(S^2 \times S^1)$ and $M = M_1 \# \ldots \# M_\ell \# m'(S^2 \times S^1)$. It follows $m' = m$.