

V B. An Algorithm to Construct the JSJ Decomposition of a 3–manifold

Abstract: This lecture will give a proof of the JSJ-Decomposition (for closed, orientable, irreducible 3–manifolds) and an algorithm for constructing the JSJ decomposition of a 3–manifold and deriving the Seifert invariants of the Characteristic Submanifold.

DEFINITIONS:

- Irreducible Decomposition
- Prime Decomposition
- Atoridal Decomposition

Definition. If \mathbb{T} is a pairwise disjoint, collection of embedded, essential tori in the 3–manifold M so that the closure of each component of $N \setminus \mathbb{T}$ is atoridal, we say \mathbb{T} is an **atoridal decomposition of N** .

- Torus Decomposition

Definition. If \mathbb{T} is the unique (up to isotopy) minimal pairwise disjoint, collection of embedded, essential tori in the 3–manifold M so that the closure of each component of $N \setminus \mathbb{T}$ is atoridal, we say \mathbb{T} is the **torus decomposition of N** .

- JSJ Decomposition

Theorem (Torus Decomposition, Jaco-Shalen, 1977; Johannson, 1977). *A closed, irreducible 3–manifold N has a unique (up to isotopy) pairwise disjoint collection of embedded, incompressible tori, \mathbb{T} , so that the closure of a component of $N \setminus \mathbb{T}$ is either atoridal or Seifert fibered.*

Proof:

Step 1. A closed, irreducible 3–manifold has an atoridal decomposition (Kneser-Haken Finiteness).

Step 2. (Existence) A closed, irreducible, orientable 3–manifold N contains a pairwise disjoint, collection of embedded, essential tori \mathbb{T} so that the closure of each component of $N \setminus \mathbb{T}$ is either atoridal or Seifert fibered.

Proof: (continued)

Step 3. (Uniqueness) There is a unique (up to isotopy) minimal pairwise disjoint, collection of embedded, essential tori \mathbb{T} so that the closure of each component of $N \setminus \mathbb{T}$ is either atoridal or Seifert fibered.

- Suppose \mathbb{T} and \mathbb{T}' are minimal collections splitting N into atoridal and Seifert components.
- If $\mathbb{T} \cap \mathbb{T}'$ is minimal (up to isotopy) for such collections, then $\mathbb{T} \cap \mathbb{T}' = \emptyset$
- Thus \mathbb{T} and \mathbb{T}' are isotopic.

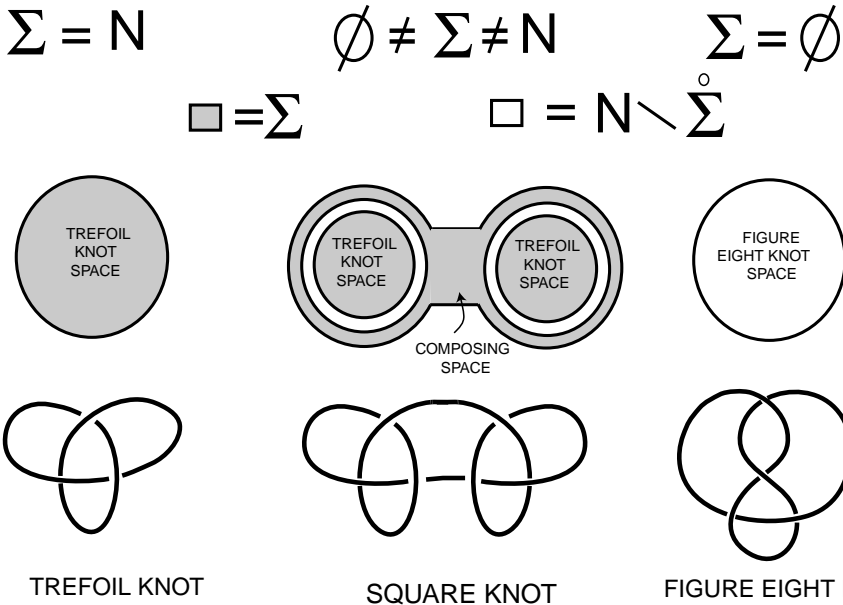
THE JSJ DECOMPOSITION

Theorem (Jaco-Shalen, 1977; Johannson, 1977).

A closed, irreducible, orientable 3-manifold M has a unique (up to isotopy) submanifold Σ so that each component of Σ is Seifert fibered and each component of $M \setminus \Sigma$ is atoroidal.

The submanifold Σ is called the **Characteristic Submanifold** of N and we write the pair as (N, Σ) .

EXAMPLES:



CONSTRUCTION OF JSJ DECOMPOSITION

Theorem (Jaco-Tollefson,1989, Jaco-Letscher-Rubinstein,1998). *Given a Haken 3–manifold N , there is an algorithm to construct the JSJ-Decomposition and the Characteristic submanifold Σ of N . Furthermore, the algorithm produces the Seifert invariants for each component of Σ .*

Proof:

The Jaco-Letscher-Rubinstein Algorithm.

1. An atoridal decomposition of N can be constructed.
 - It can be decided if a compact irreducible 3–manifold is atoridal.

- The construction uses a re-writing method to find the tori for an atoridal decomposition.
2. A closed irreducible, atoridal 3-manifold with tori boundary containing an essential annulus or disk is Seifert fibered. Furthermore, its Seifert invariants can be determined.
 - If an essential disk, then a solid torus.
 3. We construct a minimal subcollection such that each component of its complement is either Seifert fibered or atoridal. Thus we must have the unique torus decomposition.
 4. From the unique torus decomposition, we can construct the Characteristic submanifold Σ ; and hence, the JSJ decomposition.

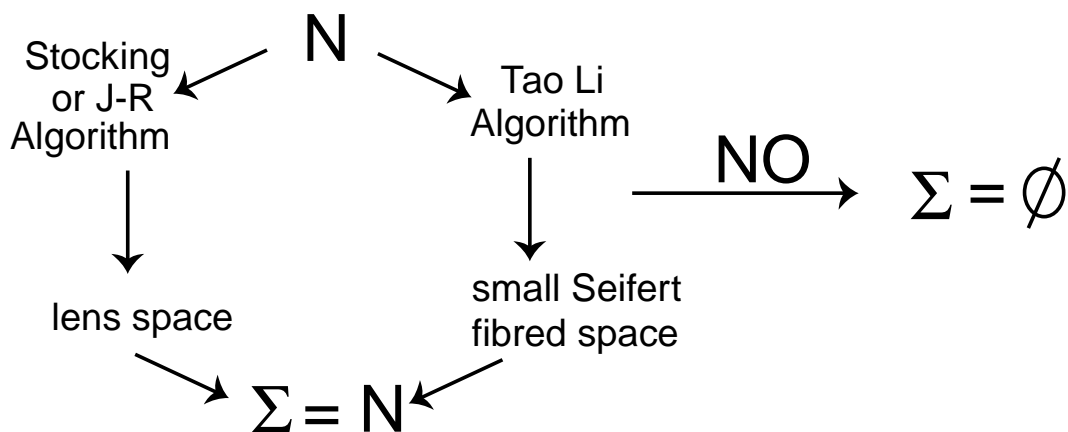
What if N has no essential tori?

For N closed, if no essential tori, then either $\Sigma = \emptyset$ or $N = \Sigma$ (N is Seifert fibered). The only non Haken, Seifert manifolds are lens spaces and small Seifert fibered spaces.

For N irreducible, we have: $\Sigma = N$; $\Sigma = \emptyset$; $N \neq \Sigma \neq \emptyset$ or N is closed, irreducible and atoroidal and we don't know if $\Sigma = N$ or $\Sigma = \emptyset$.

N closed, irreducible and atoroidal

QUESTION: Is N Seifert fibred?



Theorem (Stocking, 1998; Jaco-Rubinstein, 2001). *Given a closed, irreducible 3-manifold N , it can be decided if N is a lens space and the algorithm produces its Seifert invariants.*

Theorem (Tao Li, 2002). *Given a closed, irreducible 3-manifold N it can be decided if N is a small Seifert fibred space and the algorithm produces its Seifert invariants.*