Theorem. *A closed* 3*–manifold admits a one-vertex triangulation.*

Theorem. *A compact* 3*–manifold with boundary, no component of which is* S^2 *or* $\mathbb{R}P^2$*, admits a triangulation with all vertices in the boundary and just one vertex in each boundary component.*

Remark. The second derived of such triangulations is PL.

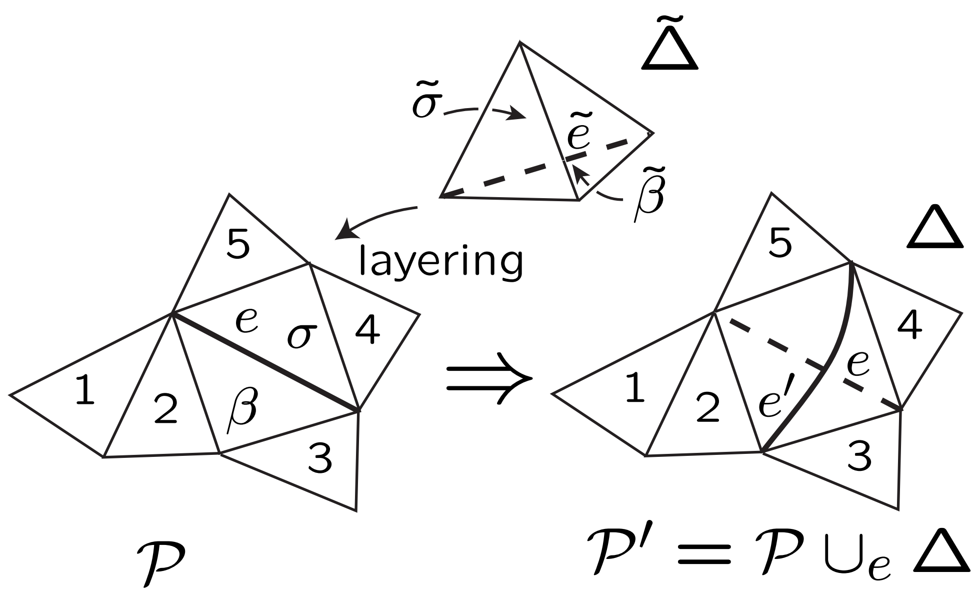
Theorem. *The interior of any compact* 3*–manifold with boundary admits an ideal triangulation.*

Definition. M a compact 3–manifold with non-empty boundary and $\mathcal{T}_\partial$ is a triangulation of ∂M. A triangulation $\mathcal{T}$ of M is an **extension of** $\mathcal{T}_\partial$ if all vertices of $\mathcal{T}$ are in ∂M and $\mathcal{T}$ induces the triangulation $\mathcal{T}_\partial$ on ∂M.

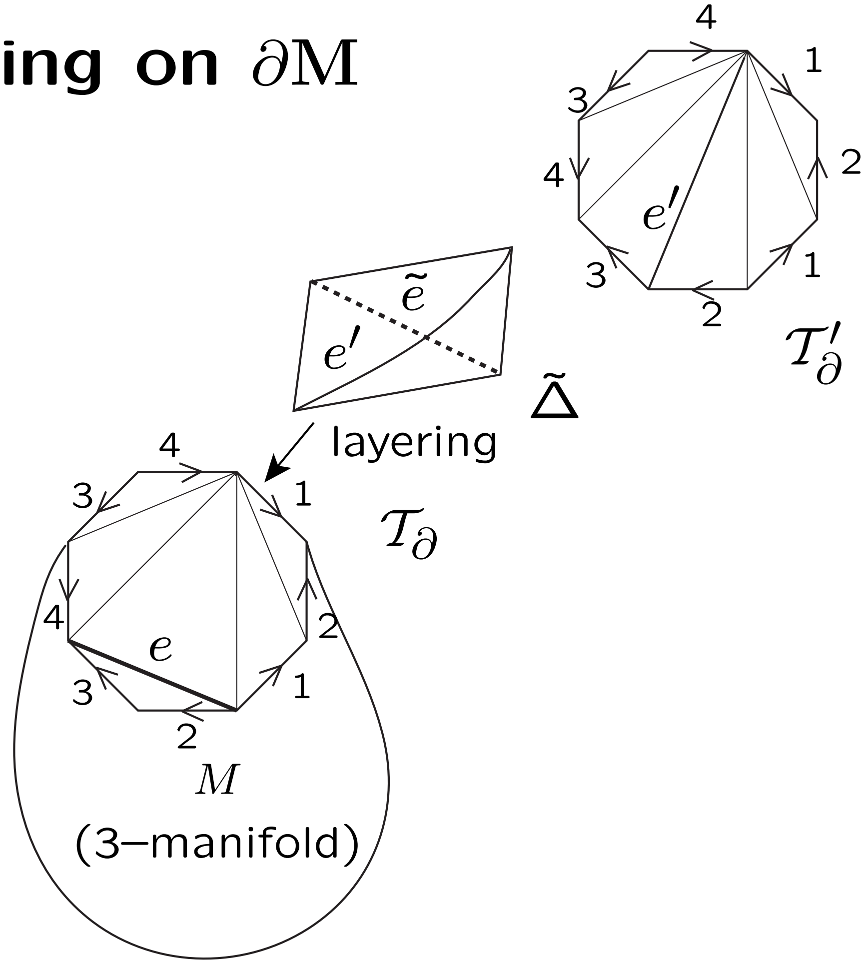
Theorem. *M a compact 3–manifold with non-empty boundary. Then any triangulation of ∂M has an extension to a triangulation of M.*

Layered Triangulations

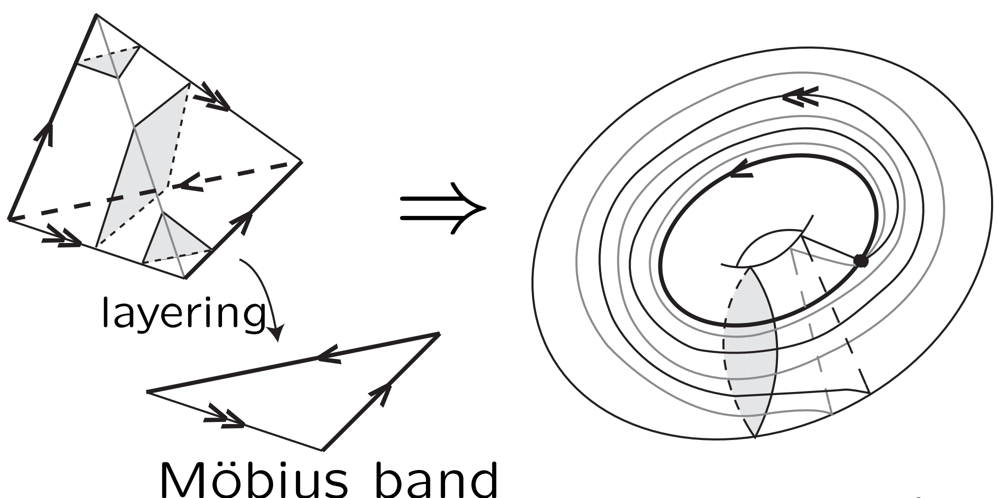
Definition. Layering a tetrahedron on a surface (or boundary of a 3–manifold).



Layering on ∂M

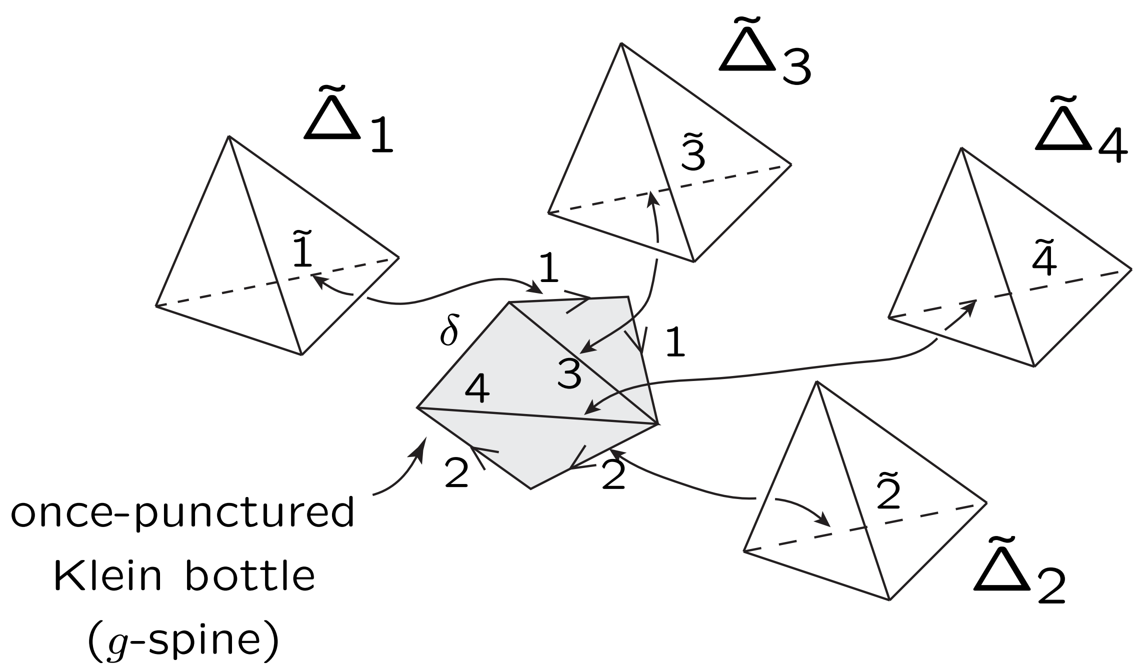


Layering on surface



• **Layered triangulations of handlebodies**

Proposition. *A minimal triangulation of a genus* g *handlebody can be obtained by layering* $3g-2$ *tetrahedra on the interior edges of a one-vertex triangulation of a surface with one boundary component and Euler characteristic* $(1-g)$ *(a* g-**spine***).*

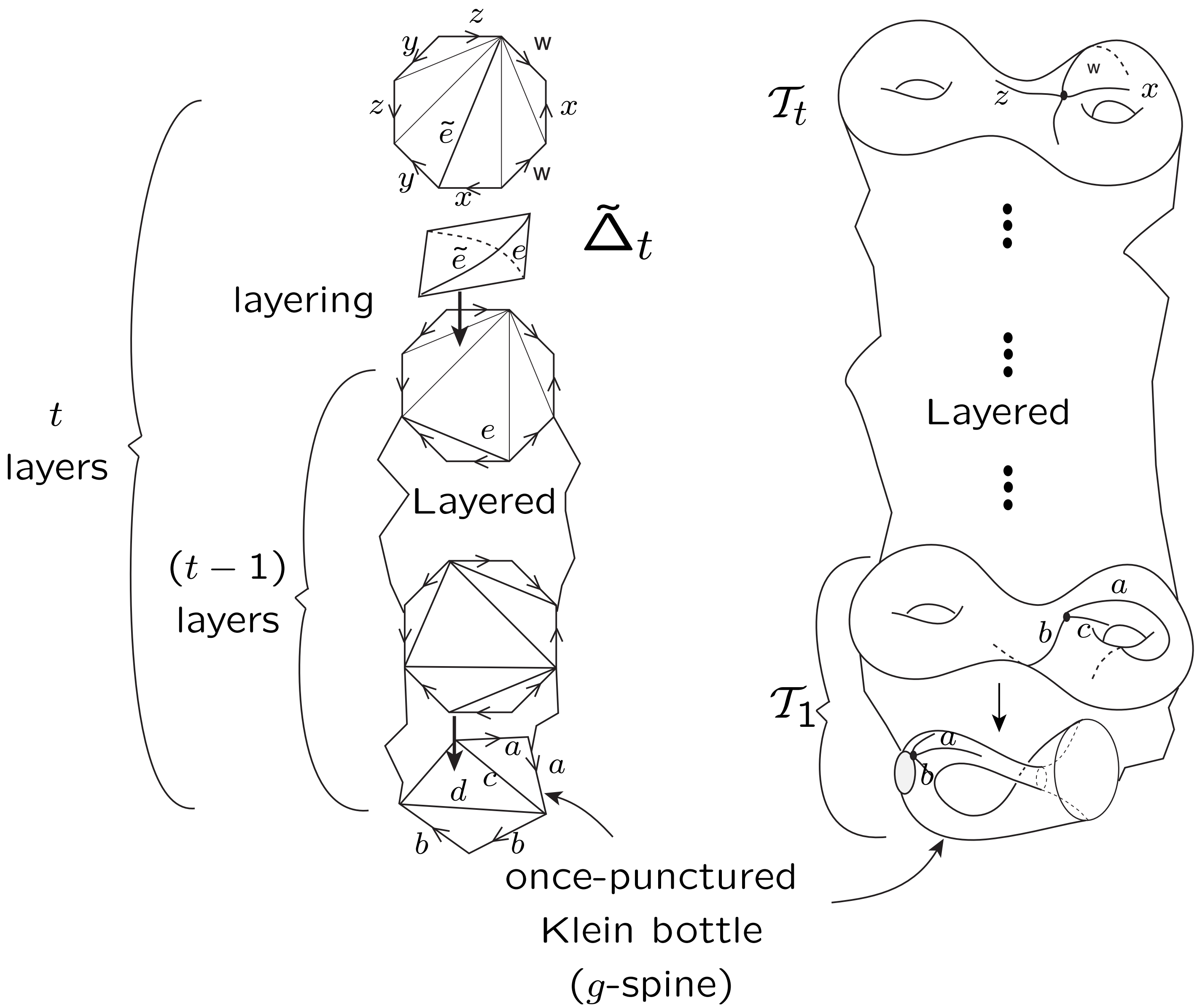


Definition. A triangulation $\mathcal{T}_t$ of a (genus g) handlebody is said to be a **layered-triangulation**, with t–layers, if

1. $\mathcal{T}_0$ is a g-spine,

2. $\mathcal{T}_1$ is a minimal layered-triangulation of the genus-g-handlebody (obtained by layering $3g-2$ tetrahedra onto the interior edges of a g-spine),

3. $\mathcal{T}_t = \mathcal{T}_{t-1} \cup_e \tilde{\Delta}_t$ is a layering along the edge e of a layered-triangulation $\mathcal{T}_{t-1}$ having $t-1$ layers, $t > 1$.


z
y
w
z
x
ẽ
y
x
w
Δ̃t
ẽ
e
layering
t
layers
e
Layered
(t − 1)
layers
a
a
c
d
b
b
once-punctured
Klein bottle
(g-spine)
𝒯t
w
z
x
Layered
a
b
c
𝒯1
a
b

Theorem. *Any one-vertex triangulation, τ, on the boundary of a handlebody has an extension to a layered-triangulation of the handlebody, called a τ–layered-triangulation.*

Definition. A τ-layered-triangulation $\mathcal{T}_\tau$ with $|\mathcal{T}_\tau| \leq |\mathcal{T}'_\tau|$ for any τ–layered-triangulation $\mathcal{T}'_\tau$ is a **minimal τ–layered-triangulation**.

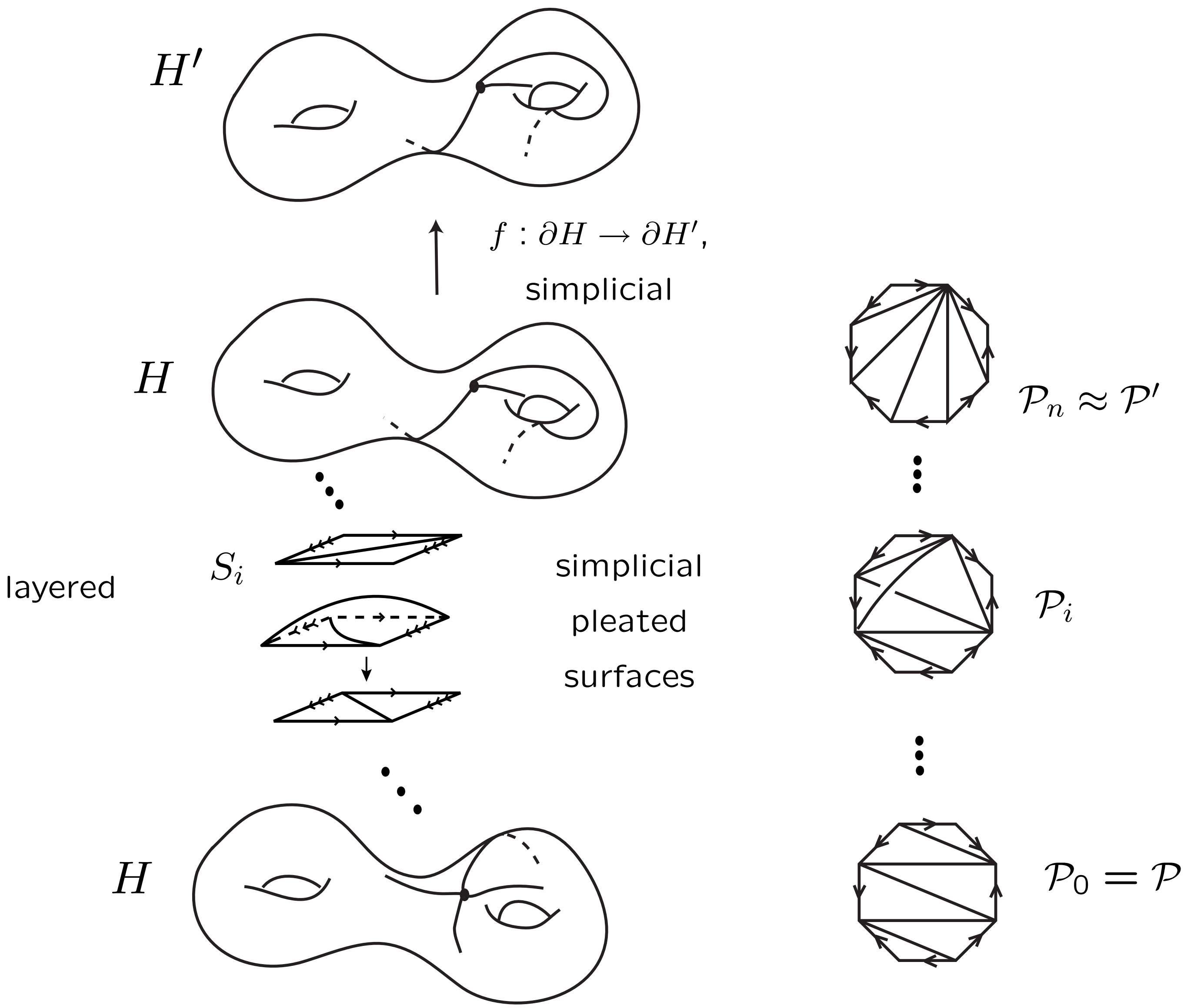
Question. *Is a minimal τ–layered-triangulation a minimal extension of the triangulation τ to a triangulation of the handlebody?*

• Layered triangulations of closed 3–manifolds

Definition. A triangulation $\mathcal{T}$ of a closed 3–manifold is a (genus g) **layered-triangulation** iff $\mathcal{T} = \mathcal{T}_H \cup \mathcal{T}_{H'}$ is the union of two subcomplexes, where $\mathcal{T}_H$ and $\mathcal{T}_{H'}$ are layered triangulations of genus g handlebodies.

Theorem. *Every closed 3–manifold admits a layered-triangulation.*

Corollary. *Every closed 3–manifold has a one-vertex triangulation.*



H'
$f : \partial H \to \partial H'$,
simplicial
H
$\mathcal{P}_n \approx \mathcal{P}'$
layered
S_i
simplicial
pleated
surfaces
$\mathcal{P}_i$
H
$\mathcal{P}_0 = \mathcal{P}$

• Triangulated Heegaard splittings

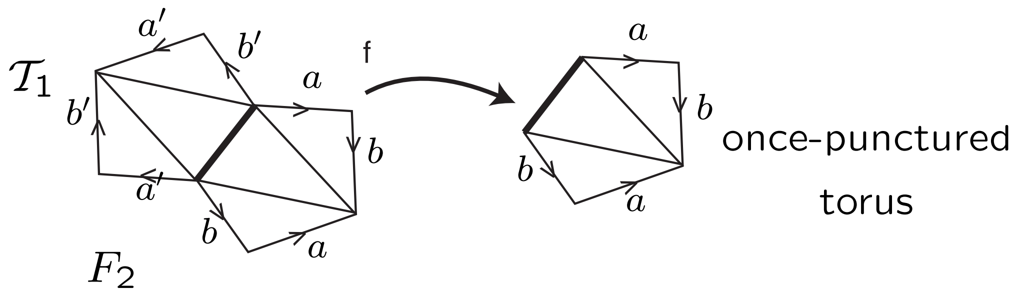
Definition. A one-vertex triangulation of a closed surface is said to be **2–symmetric** if it admits a simplicial involution fixing an edge.

EXAMPLES:

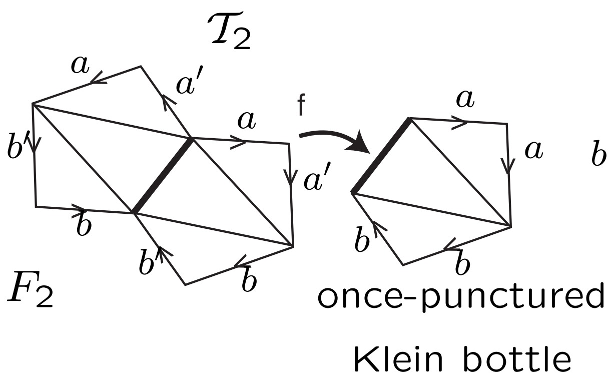
1. A one-vertex-triangulation of torus is 2–symmetric about each edge.

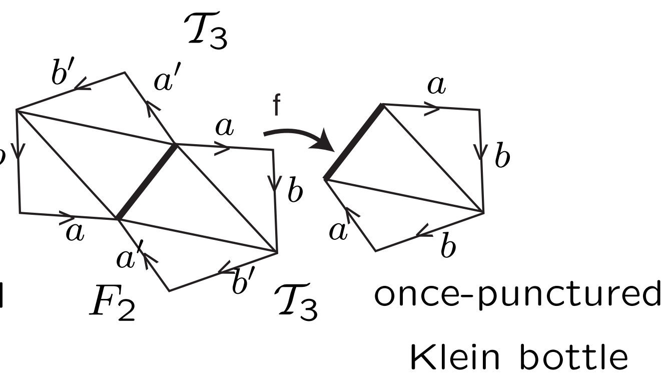
2. Five of the nine one-vertex triangulations of the genus 2 surface are 2–symmetric.

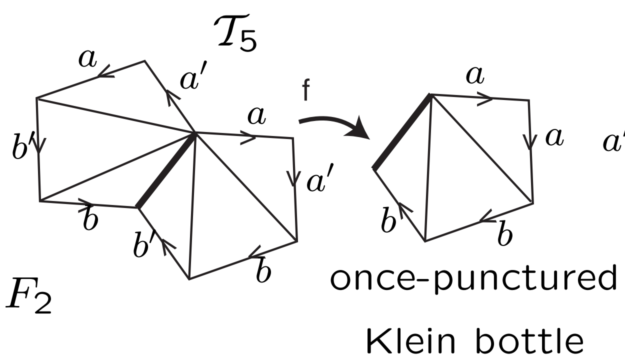
separating 2–symmetry

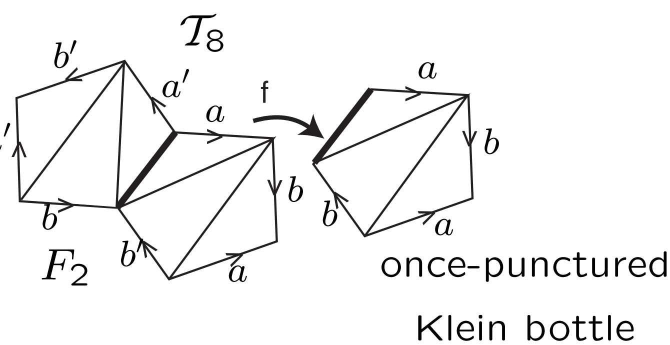


non separating 2–symmetry

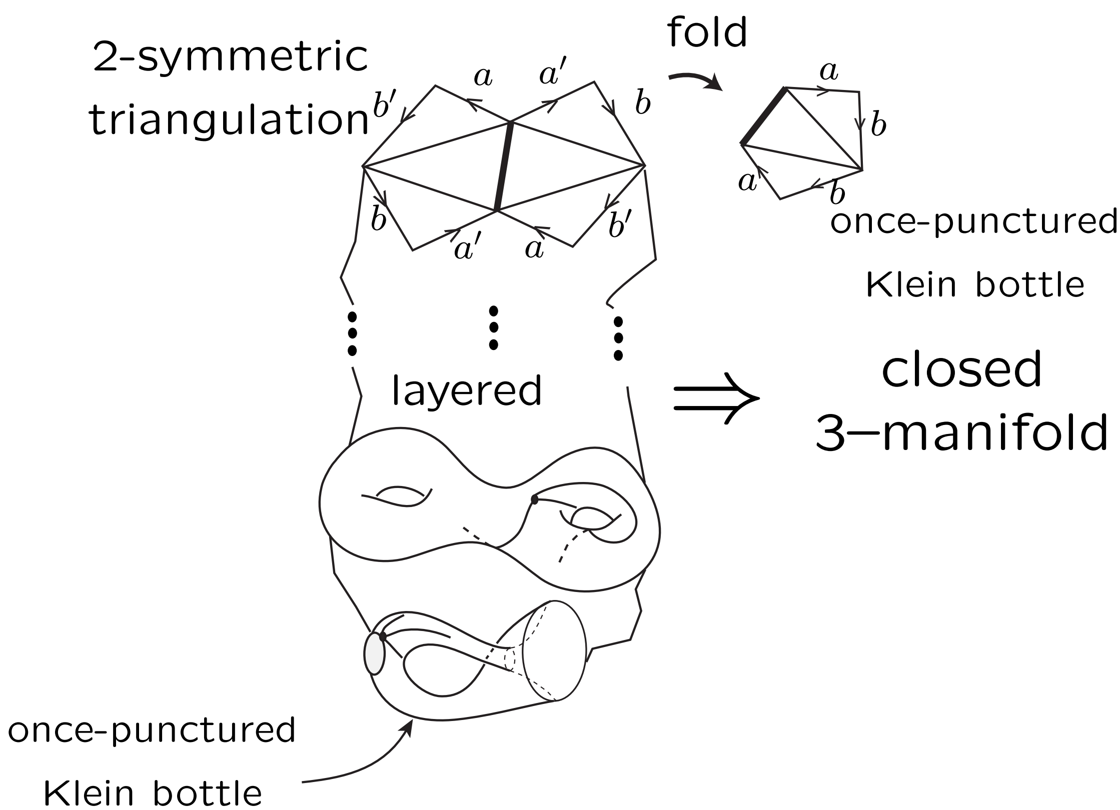






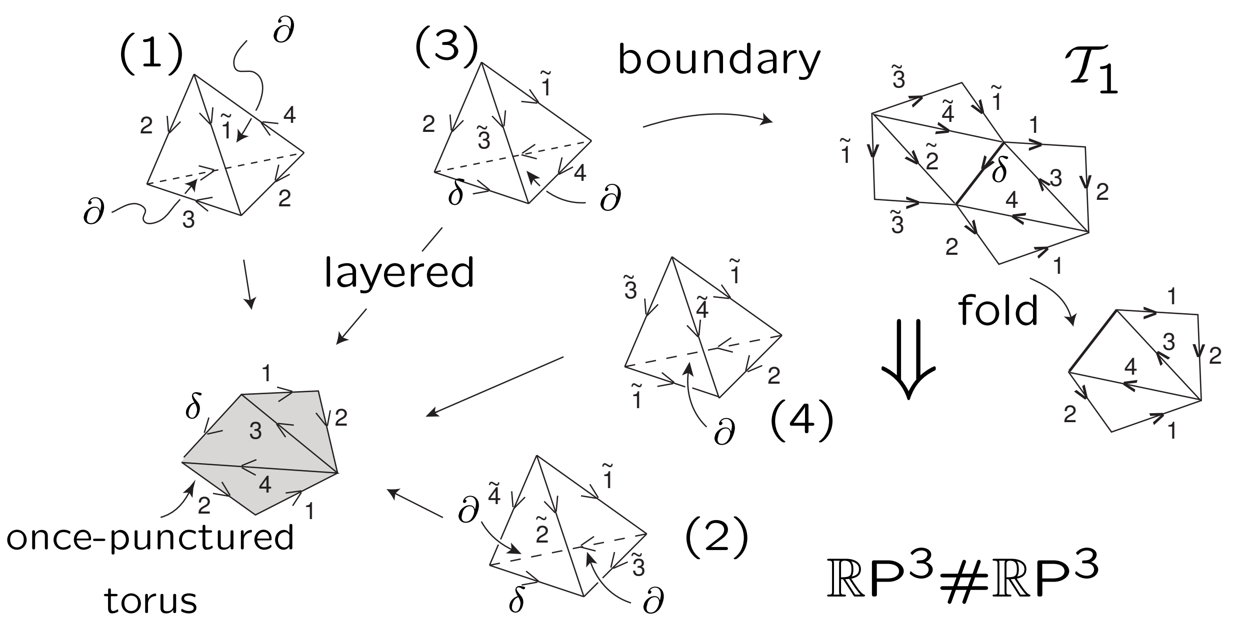


Definition. Suppose H is a genus g handlebody with a layered-triangulation and a 2–symmetric triangulation on its boundary. If M is the 3–manifold obtained by a "fold" along the edge of 2-symmetry in ∂H, then M has a layered-triangulation and we say M is presented by a **triangulated Heegaard splitting** (of genus g).

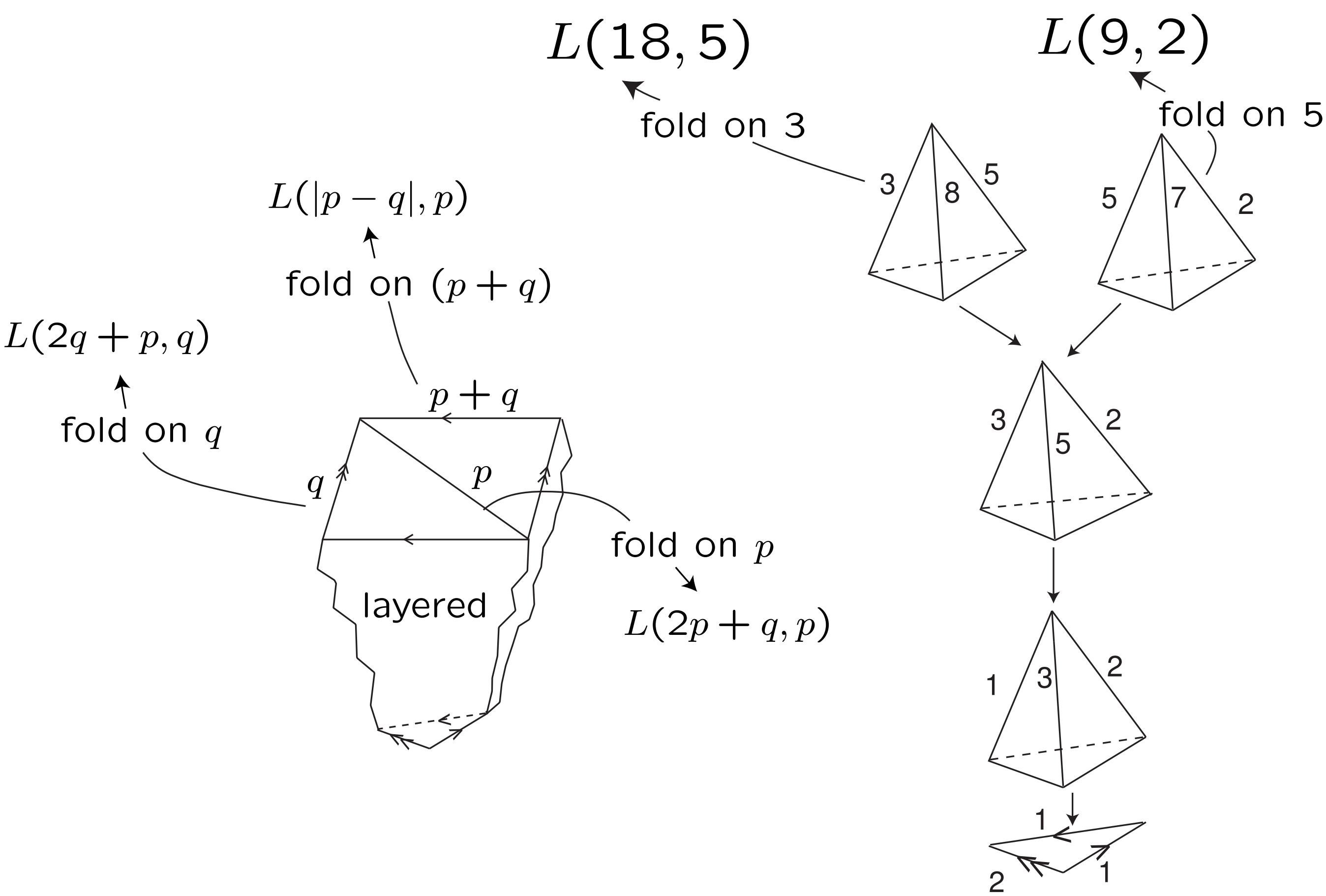


Remark. Every closed 3–manifold can be presented by a triangulated Heegaard splitting.

Example. Triangulated Heegaard splitting of $\mathbb{R}P^3 \# \mathbb{R}P^3$



Example. Triangulated Heegaard splitting of $L(18,5)$ and $L(9,2)$



Definition. Two triangulations τ, τ' on ∂H are equivalent ($\tau \sim \tau'$) iff $\exists$ a homeomorphism $h : H \to H$ with $h(\tau) = \tau'$.

Proposition. *p/q and p'/q'–triangulations on the boundary of a solid torus are equivalent iff $p/q = p'/q'$.*

$\overline{\mathbf{FLIP}}_{\mathbf{g}}$ Complex

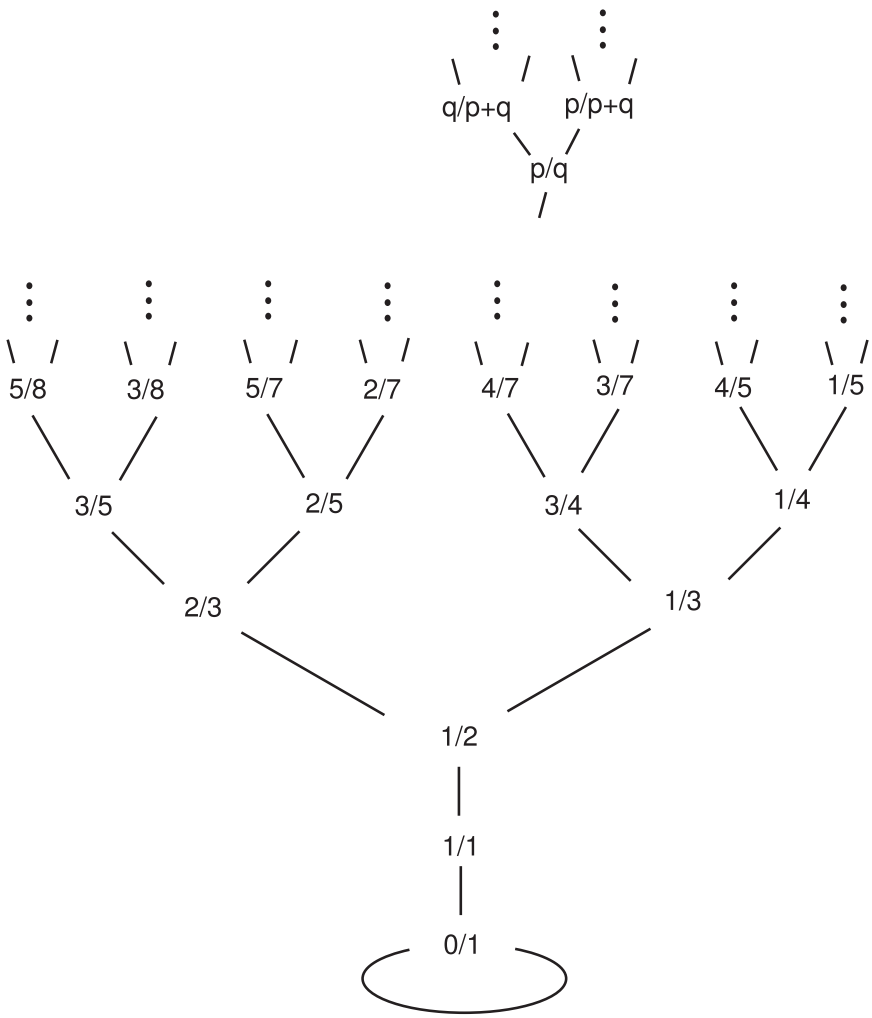
Fix a genus g handlebody H.

(0) vertices of $\overline{FLIP}_g$ are equivalence classes of one-vertex-triangulations on ∂H;

(1) edges of $\overline{FLIP}_g$ correspond to diagonal flips between triangulations.
⋮

(k) Cells can be defined up to dimension $k = 4g - 3$.

EXAMPLE: $\overline{FLIP}_1$ Complex



Applications of Layered-triangulations for lens spaces (genus 1)

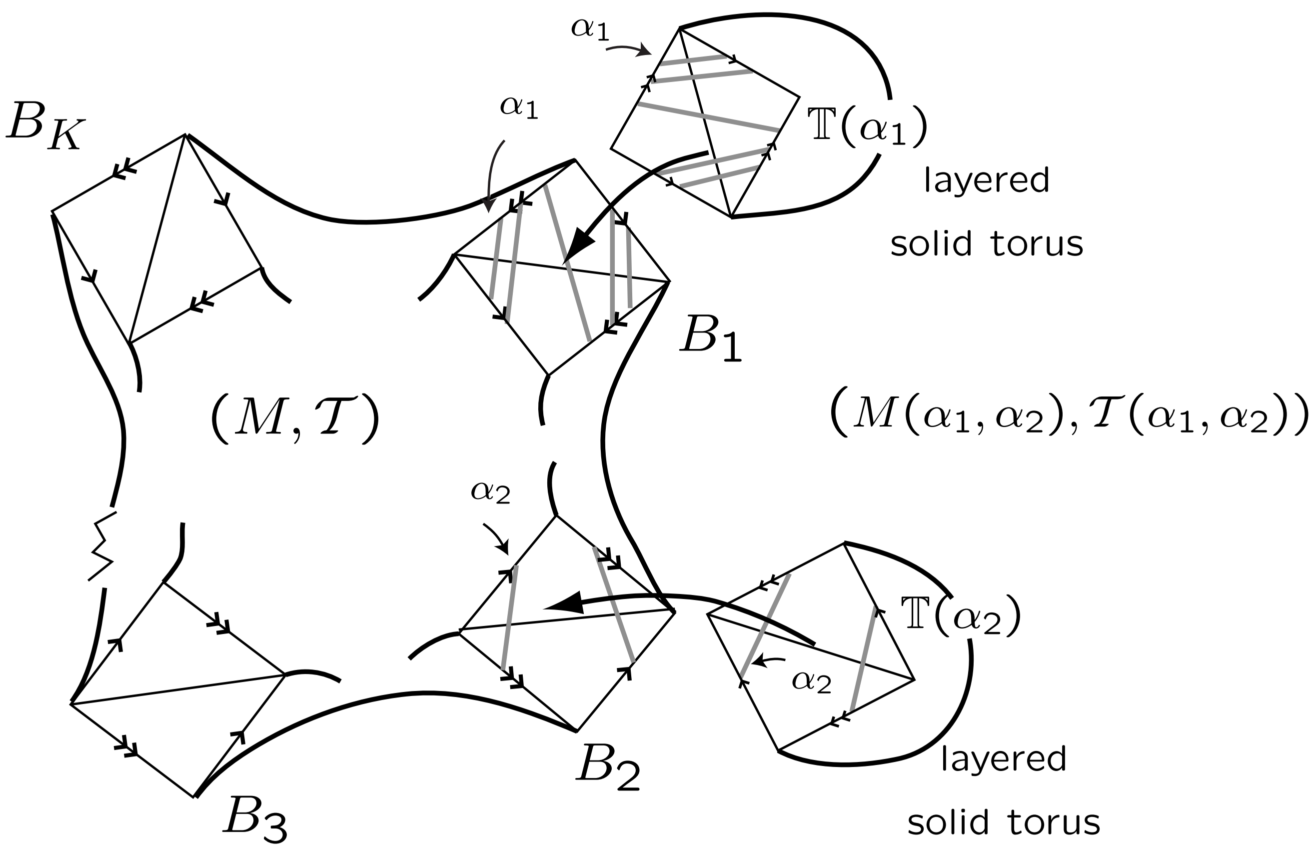
- Classification of normal and almost normal surfaces
- Classification of embedded non-orientable surfaces
- Uniqueness of Heegaard splittings
- Classification of lens spaces

Triangulated Dehn-fillings

– M, a knot- or link-manifold

– $\mathcal{T}$, a minimal vertex triangulation of M (with normal boundary)

– α_i, a slope on the torus boundary B_i



Applications of triangulated Dehn-fillings

- Decision problems in the space of Dehn-fillings

- Heegaard splittings of Dehn-fillings

- Finding planar surfaces in knot- and link-manifolds

- Efficient triangulations

Efficient triangulations

- **0–efficient triangulations**

Definition. Triangulation $\mathcal{T}$ is **0–efficient** iff

($\partial = \emptyset$) all normal 2–spheres are vertex-linking.

($\partial \neq \emptyset$) no normal 2–spheres and all normal disks are vertex-linking.

Proposition. *If M has a 0–efficient triangulation $\mathcal{T}$, then*

($\partial = \emptyset$) M is irreducible, $M \neq \mathbb{R}\mathrm{P}^3$, and $\mathcal{T}$ has either one vertex or two vertices, in which case $M = S^3$.

($\partial \neq \emptyset$) M is irreducible, ∂–irreducible, $\mathcal{T}$ has all vertices in ∂M, and either there is just one-vertex in each boundary component or $M = \mathbb{B}^3$.

Theorem. [Existence] $\exists$ *an algorithm to modify a triangulation of an irreducible, ∂–irreducible 3–manifold M to a 0–efficient triangulation or it can be shown that M is $S^3, \mathbb{R}\mathrm{P}^3$, or $L(3,1)$.*

Theorem. [Construction of Prime Decomposition] *Given a 3–manifold M with triangulation $\mathcal{T}$, there is an algorithm that constructs a decomposition*

$$M = p(S^2 \times S^1) \# q(\mathbb{R}\mathrm{P}^3) \# M_1 \# \ldots \# M_n,$$

where each M_i is given by a 0–efficient triangulation $\mathcal{T}_i$, or concludes $M = S^3$.

REMARKS:

1. $\sum |\mathcal{T}_i| \leq |\mathcal{T}|$ with equality iff $\mathcal{T}$ is 0–efficient.

2. 3–sphere recognition works for 0–efficient triangulations.

• 1–**efficient triangulations**

Definition. Triangulation $\mathcal{T}$ of a closed 3–manifold is 1–**efficient** iff it is 0–efficient and the only normal tori are edge-linking or cabled.

Proposition. *If M has a 1–efficient triangulation $\mathcal{T}$, then in addition to satisfying the conditions for a 0–efficient triangulation, M is atoridal.*

Theorem. [Existence] $\exists$ *an algorithm to modify a triangulation of an irreducible, atoridal 3–manifold M to a 1–efficient triangulation or it can be shown that M is S^3, a lens space, or a small Seifert fibre space.*

REMARK:

1. lens space recognition works for 1–efficient triangulations.

Constructions (on triangulations)

- **Crushing triangulations along normal surfaces**

- **Blow-ups of ideal triangulations**

  - edge-slopes

  - angle structures