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Complex-Valued Burgers and KdV–Burgers Equations

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Abstract Spatially periodic complex-valued solutions of the Burgers and KdV–Burgers equations are studied in this paper. It is shown that for any sufficiently large time T, there exists an explicit initial datum such that its corresponding solution of the Burgers equation blows up at T. In addition, the global convergence and regularity of series solutions is established for initial data satisfying mild conditions.

Keywords Complex Burgers equation · Complex KdV–Burgers equation · Finite-time singularity · Global regularity

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1 Introduction

This work addresses the global regularity issue on solutions of the complex Burgers and Kortweg–de Vries (KdV)–Burgers equations

$$u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, \tag{1.1}$$

where $\nu \geq 0$ and $\alpha \geq 0$ are parameters and $u = u(x,t)$ is a complex-valued function. Attention will be focused on the spatially periodic solutions, namely $x \in \mathbf{T} = \mathbf{R}/(2\pi)$, the one-torus, and we supplement (1.1) with given initial data

$$u(x,0) = u_0(x), \quad x \in \mathbf{T}. \tag{1.2}$$

Our first major result is for the complex Burgers equation ((1.1) with $\alpha = 0$), and it asserts that, for any sufficiently large time T, there exists an explicit smooth initial data u_0 such that its corresponding solution blows up at $t = T$ (Theorem 2.1). This result was partially motivated by a recent paper of Poláčik and Šverák (2008), in which the complex-valued Burgers equation on the whole line was shown to develop finite-time singularities for compactly supported smooth data. Their proof takes advantage of the explicit solution formula obtained via the Hopf–Cole transform. By contrast, the finite-time singular solutions constructed in this paper assume the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{ikx} \tag{1.3}$$

and correspond to the initial data $u_0(x) = a e^{ix}$. We emphasize that solutions of the form (1.3) are locally well posed in the usual Sobolev space $H^s := H^s(\mathbf{T})$ with a suitable index s (see Theorem 2.5 for more details). For any $T \geq T_0$ (a fixed number depending on ν only), we obtain a lower bound for $|a_k(T)|$ through a careful observation of the pattern that $a_k(t)$'s exhibit and the finite-time singularity of (1.3) in L^2 then follows if we take a in u_0 to be sufficiently large. This result reveals a fundamental difference between the real-valued solutions of the Burgers equation and their complex counterparts. The diffusion in the case of complex-valued solutions no longer dissipates the L^2-norm, which can blow up in a finite time. However, if we know that the L^2-norm of a complex-valued solution is bounded, then there would be no finite-time singularity (Theorem 2.6).

We also explore the conditions under which solutions of (1.1) are global in time. A simple example of the global solutions of (1.1) corresponds to the initial data $u_0(x) = a_0 e^{ix}$ with $|a_0| < 1$ provided ν and α satisfy a suitable condition, say $\nu^2 + 4\alpha^2 \geq 9$ (see Theorem 3.5). For general initial data of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k} e^{ikx}$$

with $|a_{0k}| < 1$, (1.1) possesses a unique local solution (1.3) with $a_k(t)$ given by a finite sum of terms that can be made explicit through an inductive relation. To show the convergence of (1.3) for large time, it is necessary to estimate $|a_k(t)|$ and our approach is to count the total number of terms that it contains. This counting problem is closely related to the number of nonnegative integer solutions to the equation

$$j_1 + 2j_2 + 3j_3 + \cdots + kj_k = k$$

for a fixed integer $k > 0$. Using a result by Hardy and Ramanujan (1918), we are able to establish the global regularity of (1.3) under a mild assumption (see Theorem 3.3). In addition, $\|u(\cdot,t)\|_{H^s}$ for any $s \geq 0$ decays exponentially in t for large t.

We remark that the study of complex-valued Burgers and KdV–Burgers equations can be justified both physically and mathematically. Physically these complex equations do arise in the modeling of several physical phenomena (Kerszberg 1984; Levi 1994; Levi and Sanielevici 1996). Mathematically these equations exhibit some remarkable features and admit solutions with much richer structures than those of their real-valued ones. In fact, these equations and other complex-valued partial differential equations have attracted quite some attention recently. Much effort has been devoted to the important issue of whether or not their solutions can blow up in a finite time. Birnir (1987) considered the complex KdV equation and constructed a family of singular solutions represented by the Weierstrass function. Very recently Li (2009) obtained simple explicit formulas for finite-time blowup solutions of the complex KdV equation through a Darboux transform. Bona and Weissler (2001) addressed the blowup issue for a family of complex-valued nonlinear dispersive equations. The papers of Wu and Yuan (2005, 2007) and Yuan and Wu (2005) treated the complex KdV and KdV–Burgers equations as systems of two nonlinearly coupled equations and clarified how the potential singularities of the real part are related to those of the imaginary part. In addition, extensive numerical experiments were performed to reveal the blowup structures. Another important example that shows significant differences between the real-valued and complex-valued solutions is the Navier–Stokes equations. It remains open as to whether or not classical solutions of the three-dimensional (3D) incompressible Navier–Stokes equations can develop finite-time singularities. However, Li and Sinai (2008) recently showed that the complex solutions of the 3D Navier–Stokes equations corresponding to large parameter family of initial data blow up in finite time. Their work motivated the study of Poláčik and Šverák on the complex-valued solutions of the Burgers equation, as we mentioned earlier.

The rest of this paper is divided into three sections. The second section focuses on the complex Burgers equation and presents Theorems 2.1, 2.5, and 2.6. The third section details the global regularity results concerning the complex KdV–Burgers equations.

2 Blow Up for the Complex Burgers Equation

This section presents three major results. The first one is a blowup result for the complex Burgers equation in a periodic domain $\mathbf{T} = [0, 2\pi]$, namely,

$$\begin{cases} u_t - 6uu_x - \nu u_{xx} = 0, & x \in \mathbf{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{T}. \end{cases} \tag{2.1}$$

It states that for any sufficiently large $T > 0$, there exists the initial data u_0 such that its corresponding solution u blows up at $t = T$. This solution can be represented by

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{ikx} \tag{2.2}$$

and the blow up is in the L^2 sense.

For the sake of completeness of our theory on (2.1), we also present a local existence and uniqueness result on solutions of the form (2.2) to the complex-valued KdV-Burgers-type equation

$$u_t - 6uu_x + \nu(-\Delta)^\gamma u + \alpha u_{xxx} = 0, \tag{2.3}$$

which reduces to the complex Burgers equation when $\gamma = 1$ and $\alpha = 0$. The fractional Laplacian $(-\Delta)^\gamma$ is defined through a Fourier transform,

$$\widehat{(-\Delta)^\gamma u}(\xi) = |\xi|^{2\gamma} \hat{u}(\xi).$$

The third result asserts that if the L^2-norm of a solution of (2.3) is bounded on $[0, T]$, then all higher derivatives are bounded and no singularity is possible on $[0, T]$.

We divide the rest of this section into two subsections with the first devoted to the blowup result and the second to the local existence uniqueness.

2.1 Finite-Time Blow Up

Theorem 2.1 *For every sufficiently large $T > 0$, there exists initial data u_0 of the form*

$$u_0(x) = a e^{ix} \tag{2.4}$$

such that the corresponding solution u of (2.1) blows up at $t = T$ in the L^2-norm, namely

$$\|u(\cdot, T)\|_{L^2(\mathbf{T})} = \infty. \tag{2.5}$$

For any $s \in \mathbf{R}$, the homogeneous Sobolev space $\mathring{H}^s(\mathbf{T})$ and the inhomogeneous Sobolev space $H^s(\mathbf{T})$ are defined in the standard fashion. In particular, a function of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{ikx}$$

is in $\mathring{H}^s(\mathbf{T})$ if

$$\|u\|^2_{\mathring{H}^s(\mathbf{T})} \equiv \sum_{k=1}^{\infty} k^{2s} |a_k|^2 < \infty,$$

and in $H^s(\mathbf{T})$ if

$$\|u\|^2_{H^s(\mathbf{T})} \equiv \sum_{k=1}^{\infty}\left(1+k^2\right)^s |a_k|^2 < \infty.$$

Clearly, $L^2(\mathbf{T})$ can be identified with $H^0(\mathbf{T})$.

For u_0 given by (2.4), the local existence and uniqueness result of the next subsection asserts that the corresponding solution u can be written as

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\mathrm{e}^{\mathrm{i}kx}$$

before it blows up. The idea is to choose large a such that

$$\left\|u(\cdot,T)\right\|^2_{L^2} = \sum_{k=1}^{\infty}\left|a_k(T)\right|^2 = \infty.$$

We attempt to find an explicit representation for $a_k(t)$. It is easy to verify the following iterative formula:

$$\begin{aligned} a_1(t) &= a\mathrm{e}^{-\nu t}, \\ a_k(t) &= 3\mathrm{i}k\mathrm{e}^{-\nu k^2 t}\int_0^t \mathrm{e}^{\nu k^2\tau}\sum_{k_1+k_2=k} a_{k_1}(\tau)a_{k_2}(\tau)\,\mathrm{d}\tau, \quad k=2,3,\ldots. \end{aligned} \tag{2.6}$$

To see the pattern in $a_k(t)$, we calculate the first few of them explicitly:

$$a_1(t) = a\mathrm{e}^{-\nu t}, \tag{2.7}$$

$$a_2(t) = -\mathrm{i}a^2\nu^{-1}\left[-3\mathrm{e}^{-2\nu t} + 3\mathrm{e}^{-4\nu t}\right], \tag{2.8}$$

$$a_3(t) = -a^3\nu^{-2}\left[9\mathrm{e}^{-3\nu t} - \frac{27}{2}\mathrm{e}^{-5\nu t} + \frac{9}{2}\mathrm{e}^{-9\nu t}\right], \tag{2.9}$$

$$a_4(t) = \mathrm{i}a^4\nu^{-3}\left[-27\mathrm{e}^{-4\nu t} + 54\mathrm{e}^{-6\nu t} - \frac{27}{2}\mathrm{e}^{-8\nu t} - 18\mathrm{e}^{-10\nu t} + \frac{9}{2}\mathrm{e}^{-16\nu t}\right], \tag{2.10}$$

$$\begin{aligned} a_5(t) = a^5\nu^{-4}\bigg[& 81\mathrm{e}^{-5\nu t} - \frac{405}{2}\mathrm{e}^{-7\nu t} + \frac{405}{4}\mathrm{e}^{-9\nu t} + \frac{135}{2}\mathrm{e}^{-11\nu t} - \frac{135}{4}\mathrm{e}^{-13\nu t} \\ & - \frac{135}{8}\mathrm{e}^{-17\nu t} + \frac{27}{8}\mathrm{e}^{-25\nu t}\bigg], \end{aligned}$$

$$\begin{aligned} a_6(t) = -\mathrm{i}a^6\nu^{-5}\bigg[& -243\mathrm{e}^{-6\nu t} + 729\mathrm{e}^{-8\nu t} - \frac{2187}{4}\mathrm{e}^{-10\nu t} - \frac{729}{4}\mathrm{e}^{-12\nu t} \\ & + 243\mathrm{e}^{-14\nu t} + \frac{81}{2}\mathrm{e}^{-18\nu t} - \frac{243}{8}\mathrm{e}^{-20\nu t} - \frac{243}{20}\mathrm{e}^{-26\nu t} + \frac{81}{40}\mathrm{e}^{-36\nu t}\bigg]. \end{aligned}$$

The following lemma summarizes the pattern exhibited by the $a_k(t)$'s.

Lemma 2.2 *For any* $t > 0$,

$$a_1(t) = ab_1(t), \qquad a_2(t) = \mathrm{i}a^2 b_2(t),$$
$$a_3(t) = -a^3 b_3(t), \qquad a_4(t) = -\mathrm{i}a^4 b_4(t) \tag{2.11}$$

and more generally, for $k = 4n + j$ *with* $n = 0, 1, 2, \dots$ *and* $j = 1, 2, 3, 4$,

$$a_k(t) = a_{4n+j}(t) = \mathrm{i}^{j-1} a^{4n+j} b_{4n+j}(t), \tag{2.12}$$

where $b_{4n+j}(t) > 0$ *for any* $t > 0$.

Remark A special consequence of this lemma is that all terms in the summation in (2.6) have the same sign, and thus,

$$\left|a_k(t)\right| = 3k\mathrm{e}^{-\nu k^2 t} \int_0^t \mathrm{e}^{\nu k^2 \tau} \sum_{k_1+k_2=k} \left|a_{k_1}(\tau)\right| \left|a_{k_2}(\tau)\right| \mathrm{d}\tau. \tag{2.13}$$

Proof of Lemma 2.2 Equation (2.12) can be shown through induction. For $n = 0$, (2.12) is just (2.11). By (2.6), $a_1(t) = a\mathrm{e}^{-\nu t}$ and

$$a_2(t) = 6\mathrm{i}a^2 \mathrm{e}^{-4\nu t} \int_0^t \mathrm{e}^{4\nu\tau} b_1^2(\tau)\,\mathrm{d}\tau = \mathrm{i}a^2 b_2(t),$$

where $b_2(t) = 6\mathrm{e}^{-4\nu t} \int_0^t \mathrm{e}^{4\nu\tau} b_1^2(\tau)\,\mathrm{d}\tau > 0$. Similarly, $a_3(t) = -a^3 b_3(t)$ and $a_4(t) = -\mathrm{i}a^4 b_4(t)$ for some $b_3(t) > 0$ and $b_4(t) > 0$.

We now consider the general case. Without loss of generality, we prove (2.12) with $k = 4n + 1$. Assume (2.12) is true for all $k < 4n + 1$. By (2.6),

$$a_k(t) = 3\mathrm{i}k\mathrm{e}^{-\nu k^2 t} \int_0^t \mathrm{e}^{\nu k^2 \tau} \sum_{k_1+k_2=k} a_{k_1}(\tau) a_{k_2}(\tau)\,\mathrm{d}\tau.$$

Noticing that $a_{k_1}(\tau)a_{k_2}(\tau)$ with $k_1 + k_2 = 4n + 1$ assumes two forms

$$a_{4n_1}(\tau)a_{4n_2+1}(\tau) \quad \text{and} \quad a_{4n_1+2}(\tau)a_{4n_2-1}(\tau)$$

where $n_1 \geq 0$, $n_2 \geq 0$ and $n_1 + n_2 = n$, we conclude by the inductive assumptions that $a_{k_1}(\tau)a_{k_2}(\tau)$ must be of the form $-\mathrm{i}a^k b_{k1,k_2}(\tau)$ for some positive function $b_{k1,k_2}(\tau) > 0$. Therefore,

$$a_k(t) = a_{4n+1}(t) = a^k b_k(t)$$

with

$$b_k(t) = 3k\mathrm{e}^{-\nu k^2 t} \int_0^t \mathrm{e}^{\nu k^2 \tau} \sum_{k_1+k_2=k} b_{k1,k_2}(\tau)\,\mathrm{d}\tau > 0 \quad \text{for any } t > 0.$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 2.1 Without loss of generality, we set $\nu = 1$. Assume

$$T \geq T_0 \equiv \sum_{k=2}^{\infty} \frac{1}{k^2} \ln \frac{3k-3}{2k-3} \tag{2.14}$$

and choose a such that

$$A \equiv a\mathrm{e}^{-T} \geq 1.$$

We prove by induction that

$$\left|a_k(T)\right| \geq A^k \quad \text{for } k = 1, 2, 3, \dots \tag{2.15}$$

which, in particular, yields (2.5). Obviously, for any $0 \leq t \leq T$,

$$\left|a_1(t)\right| \geq \left|a_1(T)\right| = a\mathrm{e}^{-T} = A \geq 1.$$

To prove (2.15) for $k \geq 2$, we recall (2.13), namely

$$\left|a_k(t)\right| = 3k\mathrm{e}^{-k^2 t} \int_0^t \mathrm{e}^{k^2\tau} \sum_{k_1+k_2=k} \left|a_{k_1}(\tau)\right|\left|a_{k_2}(\tau)\right| \mathrm{d}\tau.$$

Therefore, for $T \geq t \geq t_2 \equiv \frac{1}{4}\ln 3$,

$$\left|a_2(t)\right| = 6\mathrm{e}^{-4t} \int_0^t \mathrm{e}^{4\tau} a_1^2(\tau)\,\mathrm{d}\tau = \frac{3}{2}A^2\left(1 - \mathrm{e}^{-4t}\right) \geq A^2.$$

For $k = 3$, if $T \geq t \geq t_3 \equiv t_2 + \frac{1}{9}\ln 2$,

$$\begin{aligned}
\left|a_3(t)\right| &= 9\mathrm{e}^{-9t} \int_0^t \mathrm{e}^{9\tau} 2\left|a_1(\tau)\right|\left|a_2(\tau)\right| \mathrm{d}\tau \\
&\geq 9\mathrm{e}^{-9t} \int_{t_2}^t \mathrm{e}^{9\tau} 2\left|a_1(\tau)a_2(\tau)\right| \mathrm{d}\tau \\
&\geq 2A^3\left(1 - \mathrm{e}^{-9(t-t_2)}\right) \geq A^3.
\end{aligned}$$

More generally, for any $\geq t \geq t_k = t_{k-1} + \frac{1}{k^2}\ln\frac{3k-3}{2k-3}$,

$$\begin{aligned}
\left|a_k(t)\right| &= 3k\mathrm{e}^{-k^2 t} \int_0^t \mathrm{e}^{k^2\tau}\left(\left|a_1(\tau)\right|\left|a_{k-1}(\tau)\right| + \left|a_2(\tau)\right|\left|a_{k-2}(\tau)\right|\right. \\
&\qquad \left. + \cdots + \left|a_{k-2}(\tau)\right|\left|a_2(\tau)\right| + \left|a_{k-1}(\tau)\right|\left|a_1(\tau)\right|\right) \mathrm{d}\tau \\
&\geq 3k\mathrm{e}^{-k^2 t} \int_{t_{k-1}}^t \mathrm{e}^{k^2\tau}\left(\left|a_1(\tau)\right|\left|a_{k-1}(\tau)\right| + \left|a_2(\tau)\right|\left|a_{k-2}(\tau)\right|\right. \\
&\qquad \left. + \cdots + \left|a_{k-2}(\tau)\right|\left|a_2(\tau)\right| + \left|a_{k-1}(\tau)\right|\left|a_1(\tau)\right|\right) \mathrm{d}\tau \\
&\geq \frac{3k(k-1)}{k^2}\left(1 - \mathrm{e}^{-\nu k^2(t-t_{k-1})}\right)A^k \geq A^k.
\end{aligned}$$

If $T \geq T_0$ as defined in (2.14), then $t_k < T$ for any integer $k \geq 1$ and thus

$$\left|a_k(T)\right| \geq A^k.$$

This completes the proof of Theorem 2.1. □

We state and prove a few specific properties for $a_k(t)$.

Proposition 2.3 *Assume that u_0 is given by (2.4). For each $k \geq 1$, $a_k(t)$ is of the form*

$$a_k(t) = \sum_{m=k}^{k^2} \alpha_{k,m} \mathrm{e}^{-m\nu t} \tag{2.16}$$

where the complex-valued coefficients $\alpha_{k,m}$ satisfy

$$\sum_{m=k}^{k^2} \alpha_{k,m} = 0 \quad \text{for } k \geq 2, \tag{2.17}$$

$$\alpha_{k,m} = \frac{3\mathrm{i}k}{k^2 - m} \sum_{k_1+k_2=k} \sum_{m_1+m_2=m} \alpha_{k_1,m_1}\alpha_{k_2,m_2} \quad \text{for } k \leq m < k^2. \tag{2.18}$$

The indices k_1, k_2, m_1 and m_2 in the summation above obey

$$1 \leq k_1 \leq k-1, \quad 1 \leq k_2 \leq k-1, \quad k_1 \leq m_1 \leq k_1^2 \quad \text{and} \quad k_2 \leq m_2 \leq k_2^2.$$

Proof Equation (2.17) is a consequence of the fact that $a_k(0) = 0$ for $k \geq 2$. Equation (2.16) follows from a simple induction. Obviously, $a_1(t) = a\mathrm{e}^{-\nu t}$. Fix k and assume that (2.16) is valid for all integers up to k. Then, for $k_1 \geq 1$, $k_2 \geq 1$, $k_1 + k_2 = k+1$, $k_1 \leq m_1 \leq k_1^2$ and $k_2 \leq m_2 \leq k_2^2$,

$$\begin{aligned} a_{k+1}(t) &= 3\mathrm{i}(k+1) \sum_{k_1+k_2=k+1} \sum_{m_1,m_2} \alpha_{k_1,m_1}\alpha_{k_2,m_2} \mathrm{e}^{-\nu(k+1)^2 t} \\ &\quad \times \int_0^t \mathrm{e}^{\nu((k+1)^2-(m_1+m_2))\tau}\, \mathrm{d}\tau \\ &= \sum_{k_1+k_2=k+1} \sum_{m_1,m_2} \frac{3i(k+1)\alpha_{k_1,m_1}\alpha_{k_2,m_2}}{\nu((k+1)^2-(m_1+m_2))} \left(\mathrm{e}^{-\nu(m_1+m_2)t} - \mathrm{e}^{-\nu(k+1)^2 t}\right). \end{aligned}$$

Since $m_1 + m_2 \leq k_1^2 + k_2^2 \leq (k_1+k_2)^2 = (k+1)^2$, this proves (2.16) with (2.18). □

Proposition 2.4 *Assume that u_0 is given by (2.4).*

(1) *Let $k \geq 1$ be an integer. Then*

$$\alpha_{k,k} = \left(\frac{3\mathrm{i}}{\nu}\right)^{k-1} a^k \quad \text{and} \quad \alpha_{k,k+2} = -\frac{k}{2}\alpha_{k,k}; \tag{2.19}$$

(2) *Let* $k \geq 1$ *be an integer. Then, for* $n = 1, 3, 5, \ldots,$

$$\alpha_{k,k+n} = 0;$$

(3) *Let* $k \geq 1$ *be an integer and let* $k^2 > m > U(k) \equiv k^2 - 2k + 2$. *Then*

$$\alpha_{k,m} = 0. \tag{2.20}$$

Proof Letting $m_1 = k_1$ and $m_2 = k_2$ in (2.18), we find

$$\alpha_{k,k} = \sum_{k_1+k_2=k} \alpha_{k_1,k_1}\alpha_{k_2,k_2}\frac{3ik}{\nu(k^2-k)} = \frac{3i}{\nu(k-1)}\sum_{k_1+k_2=k}\alpha_{k_1,k_1}\alpha_{k-k_1,k-k_1}.$$

A simple induction allows us to obtain the expression for $\alpha_{k,k}$. To show $\alpha_{k,k+2} = -\frac{k}{2}\alpha_{k,k}$, we set $m = k+2$ in (2.18) to get

$$\begin{aligned}\alpha_{k,k+2} = \frac{3ik}{\nu(k^2-k-2)}(&\alpha_{1,1}\alpha_{k-1,k+1} + \alpha_{2,2}\alpha_{k-2,k} + \alpha_{2,4}\alpha_{k-2,k-2} \\ &+ \cdots + \alpha_{k-2,k-2}\alpha_{2,4} + \alpha_{k-2,k}\alpha_{2,2} + \alpha_{1,1}\alpha_{k-1,k+1}).\end{aligned} \tag{2.21}$$

Inserting the inductive assumptions such as

$$\alpha_{k-1,k+1} = -\frac{k-1}{2}\alpha_{k-1,k-1}, \qquad \alpha_{k-2,k} = -\frac{k-2}{2}\alpha_{k-2,k-2}, \qquad \alpha_{2,4} = -\alpha_{2,2}$$

in (2.21), we obtain

$$\begin{aligned}\alpha_{k,k+2} &= \frac{3ik}{\nu(k^2-k-2)}\left[-\frac{k}{2}\sum_{k_1=1}^{k-1}\alpha_{k_1,k_1}\alpha_{k-k_1,k-k_1} + \alpha_{1,1}\alpha_{k-1,k-1}\right] \\ &= -\frac{k}{2}\frac{k^2-k}{k^2-k-2}\frac{3ik}{\nu(k^2-k)}\sum_{k_1=1}^{k-1}\alpha_{k_1,k_1}\alpha_{k-k_1,k-k_1} \\ &\quad + \frac{3ik}{\nu(k^2-k-2)}\alpha_{1,1}\alpha_{k-1,k-1} \\ &= -\frac{k}{2}\frac{k^2-k}{k^2-k-2}\alpha_{k,k} - \frac{k}{2}\frac{-2}{k^2-k-2}\alpha_{k,k} = -\frac{k}{2}\alpha_{k,k}.\end{aligned}$$

To show $\alpha_{k,k+1} = 0$, we set $m = k+1$ to obtain

$$\alpha_{k,k+1} = \frac{3ik}{\nu(k^2-(k+1))}(\alpha_{1,1}\alpha_{k-1,k} + \alpha_{2,2}\alpha_{k-2,k-1} + \cdots + \alpha_{k-1,k}\alpha_{1,1}),$$

which can be seen to be zero after inserting the inductive assumptions.

To prove (2.20), it suffices to notice in (2.18) that the second summation is over $m_1 + m_2 = m$ with $k_1 \leq m_1 \leq k_1^2$ and $k_2 \leq m_2 \leq k_2^2$. Thus, $m = m_1 + m_2 \leq k_1^2 + k_2^2 = (k_1+k_2)^2 - 2k_1k_2 \leq k^2 - 2(k-1)$ and $\alpha_{k,m}$ with $U(k) < m < k^2$ is equal to zero. This completes the proof of Proposition 2.4. □

2.2 Local Well-Posedness

This subsection establishes the following two major results.

Theorem 2.5 *Consider (2.3) with $\gamma > \frac{1}{2}$. Let $s > \frac{1}{2}$. Assume $u_0 \in H^s(\mathbf{T})$ has the form*

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k} \mathrm{e}^{\mathrm{i}kx}. \tag{2.22}$$

Then there exists $T = T(\|u_0\|_{H^s})$ such that (2.3) with the initial data u_0 has a unique solution $u \in C([0,T); H^s) \cap L^2([0,T); \mathring{H}^{s+\gamma})$ that assumes the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \mathrm{e}^{\mathrm{i}kx}.$$

In the case when $\gamma \geq 1$, we can actually show that no finite-time singularity is possible if we know that the L^2-norm is bounded a priori. In fact, the following theorem states that the L^2-norm controls all higher-order derivatives.

Theorem 2.6 *Let $T > 0$ and let u be a weak solution of (2.3) with $\gamma \geq 1$ on the time interval $[0,T]$. If we know a priori that $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \mathring{H}^\gamma)$, namely,*

$$M_0 \equiv \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^\gamma u(\cdot,t)\|_{L^2}^2 \,\mathrm{d}t < \infty, \tag{2.23}$$

then, for any integer $k > 0$,

$$M_k \equiv \sup_{t\in[0,T]} \|u^{(k)}(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{k+\gamma} u(\cdot,t)\|_{L^2}^2 \,\mathrm{d}t < \infty,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $u^{(k)}$ denotes any partial derivative of order k.

We first prove Theorem 2.5.

Proof of Theorem 2.5 The existence of such a solution follows from the Galerkin approximation. Let $N \geq 1$ and denote by P_N the projection on the subspace $\{\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{2\mathrm{i}x}, \dots, \mathrm{e}^{\mathrm{i}Nx}\}$. Let

$$u^N(x,t) = \sum_{k=1}^{N} a_k^N(t) \mathrm{e}^{\mathrm{i}kx}$$

where $a_k(t)$ satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} a_k^N(t) = 3\mathrm{i}k \sum_{k_1+k_2=k} a_{k_1}^N(t) a_{k_2}^N(t) + \mathrm{i}\alpha k^3 a_k^N(t) - \nu k^{2\gamma} a_k^N(t), \\ &a_k^N(0) = a_{0k}^N \equiv a_{0k}. \end{aligned} \tag{2.24}$$

Here $1 \le k_1 \le N$ and $1 \le k_2 \le N$. From the theory of ordinary differential equations, we know that (2.24) has a unique local solution $a_k^N(t)$ on $[0, T]$. We derive some a priori bounds for $u^N(x,t)$. Clearly, $u^N(x,t)$ solves

$$\partial_t u^N = 6P_N\left(u^N u_x^N\right) + \alpha u_{xxx}^N - \nu(-\Delta)^\gamma u^N, \qquad u^N(x,0) = P_N u_0.$$

We now show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^N\right\|_{H^s}^2 + \nu\left\|u^N\right\|_{H^{s+\gamma}}^2 \le C(\nu,s)\left\|u^N\right\|_{H^s}^{\frac{6\gamma-2}{2\gamma-1}}. \tag{2.25}$$

It follows from the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}a_k^N(t) + \nu k^{2\gamma}a_k^N(t) - \mathrm{i}\alpha k^3 a_k^N(t) = 3\mathrm{i}k\sum_{k_1+k_2=k} a_{k_1}^N(t)a_{k_2}^N(t)$$

that, after omitting the upper index N for notational convenience,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^N k^{2s}\left|a_k(t)\right|^2 = -2\nu\sum_{k=1}^N k^{2(s+\gamma)}\left|a_k(t)\right|^2 - 6\sum_{k=1}^N k^{2s+1}\mathcal{I}\left(\bar{a}_k\sum_{k_1+k_2=k} a_{k_1}a_{k_2}\right),$$

where $\mathcal{I}$ denotes the imaginary part. To bound the nonlinear term on the right (denoted by J), we first notice that the summation over $k_1 + k_2 = k$ is less than twice the summation over $k_1 + k_2 = k$ with $k_1 \le k_2$ and $2k_2 \ge k$. Thus,

$$\begin{aligned}
J &\le 6\sum_{k=1}^N k^{2s+1}|a_k|\sum_{k_1+k_2=k}|a_{k_1}||a_{k_2}| \\
&\le 12\sum_{k=1}^N k^{s+\frac{1}{2}}|a_k|\sum_{k/2\le k_2\le k}(2k_2)^{s+\frac{1}{2}}|a_{k_1}||a_{k_2}|.
\end{aligned}$$

Applying Hölder's inequality and Young's inequality for series, we have

$$\begin{aligned}
J &\le 12\left[\sum_{k=1}^N k^{2s+1}|a_k|^2\right]^{\frac{1}{2}}\left[\sum_{k=1}^N\left(\sum_{k/2\le k_2\le k}(2k_2)^{s+\frac{1}{2}}|a_{k_1}||a_{k_2}|\right)^2\right]^{\frac{1}{2}} \\
&\le 12\left[\sum_{k=1}^N k^{2s+1}|a_k|^2\right]^{\frac{1}{2}}\left[\sum_{k_2=1}^N k_2^{2s+1}|a_{k_2}|^2\right]^{\frac{1}{2}}\sum_{k_1=1}^N|a_{k_1}| \\
&\le 12\sum_{k=1}^N k^{2s+1}|a_k|^2\left[\sum_{k_1=1}^N|k_1|^{2s}|a_{k_1}|\right]^{\frac{1}{2}}\left[\sum_{k_1=1}^N k_1^{-2s}\right]^{\frac{1}{2}} \\
&\le C(s)\left\|u^N\right\|_{\mathring{H}^{s+\frac{1}{2}}}^2\left\|u^N\right\|_{H^s}.
\end{aligned} \tag{2.26}$$

Thus, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^N\|_{H^s}^2 + 2\nu\|u^N\|_{\mathring{H}^{s+\gamma}}^2 \le C(s)\|u^N\|_{\mathring{H}^{s+\frac{1}{2}}}^2\|u^N\|_{H^s}. \tag{2.27}$$

By Hölder's inequality

$$\|u^N\|_{\mathring{H}^{s+\frac{1}{2}}} \le \|u^N\|_{\mathring{H}^{s+\gamma}}^{\frac{1}{2\gamma}}\|u^N\|_{H^s}^{1-\frac{1}{2\gamma}},$$

we have

$$J \le C(s)\|u^N\|_{\mathring{H}^{s+\gamma}}^{\frac{1}{\gamma}}\|u^N\|_{H^s}^{3-\frac{1}{\gamma}} \le \nu\|u^N\|_{\mathring{H}^{s+\gamma}}^2 + C(\nu,s)\|u^N\|_{H^s}^{\frac{6\gamma-2}{2\gamma-1}}. \tag{2.28}$$

Equation (2.27) and (2.28) yield (2.25). With these bounds at our disposal, the existence of a solution u of the form (2.2) is then obtained as a limit of u^N as $N\to\infty$.

We now turn to the uniqueness. Assume (2.3) has two solutions u_1 and u_2 satisfying

$$u_1, u_2 \in C\big([0,T); H^s\big)\cap L^2\big([0,T); \mathring{H}^{s+\gamma}\big).$$

Then their difference $w = u_1 - u_2$ satisfies

$$w_t + \nu(-\Delta)^\gamma w + \alpha w_{xxx} = 6wu_{1x} + 6u_2 w_x.$$

Applying the same procedure as in the derivation of (2.27), we find that, for $s > \frac{1}{2}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^s}^2 + 2\nu\|w\|_{\mathring{H}^{s+\gamma}}^2 \le C(s)\|w\|_{H^s}^2\big(\|u_1\|_{\mathring{H}^1} + \|u_2\|_{\mathring{H}^1}\big).$$

The fact that $u_1, u_2 \in L^2([0,T); \mathring{H}^{s+\gamma})$ with $s+\gamma > 1$ and an application of Gronwall's inequality yield the uniqueness. This completes the proof of Theorem 2.5. □

Proof of Theorem 2.6 We start with the case $k = 1$. It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(\cdot,t)\|_{L^2}^2 + 2\kappa\|\Lambda^\gamma u_x\|_{L^2}^2 = I_1 + I_2, \tag{2.29}$$

where

$$I_1 = 2\int |u_x|^2\mathcal{R}(u_x)\,\mathrm{d}x,$$

$$I_2 = 2\int \mathcal{R}(u\bar{u}_x u_{xx})\,\mathrm{d}x.$$

Here $\mathcal{R}$ denotes the real part. By the Gagliardo–Nirenberg-type equalities,

$$\begin{aligned}|I_1| &\le 2\|u_x\|_{L^2}^2\|u_x\|_{L^\infty}\\ &\le C\|u\|_{L^2}^{\gamma_1}\|u_x\|_{L^2}^2\|\Lambda^{1+\gamma}u\|_{L^2}^{1-\gamma_1},\end{aligned}$$

$$
\begin{aligned}
|I_2| &\le C\|u\|_{L^\infty}\|u_x\|_{L^2}\|u_{xx}\|_{L^2} \\
&\le C\|u\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^{\frac{3}{2}}\|u_{xx}\|_{L^2} \\
&\le C\|u\|_{L^2}^{\frac{1}{2}+\gamma_2}\|u_x\|_{L^2}^{\frac{3}{2}}\left\|\Lambda^{1+\gamma}u\right\|_{L^2}^{1-\gamma_2}
\end{aligned}
$$

where

$$\gamma_1 = \frac{2\gamma-1}{2\gamma+2} \quad \text{and} \quad \gamma_2 = \frac{\gamma-1}{\gamma+1}.$$

By Young's inequality,

$$
\begin{aligned}
|I_1| &\le \frac{\nu}{2}\left\|\Lambda^{1+\gamma}u\right\|_{L^2}^2 + C\nu^{-\frac{1-\gamma_1}{1+\gamma_1}}\|u\|_{L^2}^{\frac{2\gamma_1}{1+\gamma_1}}\|u_x\|_{L^2}^{\frac{4}{1+\gamma_1}}, \\
|I_2| &\le \frac{\nu}{2}\left\|\Lambda^{1+\gamma}u\right\|_{L^2}^2 + C\nu^{-\frac{1-\gamma_2}{1+\gamma_2}}\|u\|_{L^2}^{\frac{1+2\gamma_2}{1+\gamma_2}}\|u_x\|_{L^2}^{\frac{3}{1+\gamma_2}}.
\end{aligned}
$$

Inserting these inequalities in (2.29) and integrating with respect to t yields

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\|u_x(\cdot,t)\right\|_{L^2}^2 + \nu\int_0^T \left\|\Lambda^{1+\gamma}u\right\|_{L^2}^2 \,\mathrm{d}t \\
&\quad\le C(\nu)M_0^{\frac{\gamma_1}{1+\gamma_1}}\int_0^T \|u_x\|_{L^2}^{\frac{4}{1+\gamma_1}}\,\mathrm{d}t + C(\nu)M_0^{\frac{1+2\gamma_2}{2+2\gamma_2}}\int_0^T \|u_x\|_{L^2}^{\frac{3}{1+\gamma_2}}\,\mathrm{d}t,
\end{aligned}
$$

where M_0 is specified in (2.23). By (2.23) and the Gagliardo–Nirenberg-type inequality

$$\|u_x\|_{L^2} \le C\|u\|_{L^2}^{1-\frac{1}{\gamma}}\left\|\Lambda^\gamma u\right\|_{L^2}^{\frac{1}{\gamma}},$$

we have

$$\int_0^T \|u_x\|_{L^2}^{2\gamma}\,\mathrm{d}t \le CM_0^\gamma.$$

Therefore,

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\|u_x(\cdot,t)\right\|_{L^2}^2 + \nu\int_0^T \left\|\Lambda^{1+\gamma}u\right\|_{L^2}^2 \,\mathrm{d}t \\
&\quad\le C(\nu)M_0^{\frac{4\gamma^2+3\gamma-1}{4\gamma+1}}\sup_{t\in[0,T]}\left\|u_x(\cdot,t)\right\|_{L^2}^{\frac{-8\gamma^2+6\gamma+8}{4\gamma+1}} \\
&\qquad + C(\nu)M_0^{\frac{4\gamma^2+3\gamma-1}{4\gamma}}\sup_{t\in[0,T]}\left\|u_x(\cdot,t)\right\|_{L^2}^{\frac{-4\gamma^2+3\gamma+3}{2\gamma}}.
\end{aligned}
\tag{2.30}
$$

When $\gamma > \frac{3}{4}$, $4\gamma^2+\gamma-3>0$ and consequently

$$\frac{-8\gamma^2+6\gamma+8}{4\gamma+1} < 2 \quad \text{and} \quad \frac{-4\gamma^2+3\gamma+3}{2\gamma} < 2.$$

Equation (2.30) then implies that

$$\sup_{t\in[0,T]} \left\|u_x(\cdot,t)\right\|_{L^2}^2 + \nu \int_0^T \left\|\Lambda^{1+\gamma} u\right\|_{L^2}^2 \mathrm{d}t \le M_1,$$

where M_1 is a constant depending on γ, ν and M_0 alone. L^2-bounds for higher-order derivatives can be obtained through iteration. This completes the proof of Theorem 2.6. □

3 Global solutions of the complex KdV–Burgers equation

We consider the initial value problem for the complex KdV–Burgers equation

$$\begin{cases} u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, & x \in \mathbf{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{T} \end{cases} \tag{3.1}$$

and study the global regularity of its solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \mathrm{e}^{\mathrm{i}kx}. \tag{3.2}$$

Here $\alpha \ge 0$ and $\nu \ge 0$ and (3.1) includes the complex Burgers and complex KdV equations as special cases. Two major results are established. Theorem 3.3 presents a general conditional global regularity result, and Theorem 3.5 asserts the global regularity of (3.2) for a special case.

Assume the initial data u_0 is of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k} \mathrm{e}^{\mathrm{i}kx} \tag{3.3}$$

and is in H^s with $s > \frac{1}{2}$. According to Theorem 2.5, (3.1) has a unique local solution $u \in C([0,T); H^s)$ of the form (3.2) for some $T > 0$. To study the global regularity of (3.2), we explore the structure of $a_k(t)$ and obtain the following two propositions.

Proposition 3.1 *If (3.2) solves (3.1), then $a_k(t)$ can be written as*

$$a_k(t) = \sum_{k\le h\le k^2, k\le l\le k^3} a_{k,h,l} \mathrm{e}^{-(\nu h - \alpha i l)t} \tag{3.4}$$

where $a_{k,h,l}$ consists of a finite number of terms of the form

$$C(\alpha,\nu,k,h,l,j_1,\dots,j_k) a_{01}^{j_1} a_{02}^{j_2} \cdots a_{0k}^{j_k} \tag{3.5}$$

with $j_1, j_2, \dots, j_k$ being nonnegative integers and satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k. \tag{3.6}$$

Proposition 3.2 *Let $k \ge 1$ be an integer. Let $U(k) = k^2 - 2k + 2$ and $V(k) = k^3 - 3k^2 + 3k$. The coefficients $a_{k,h,l}$ in (3.4) have the following properties*:

(1) *For $k \le h < k^2$ and $k \le l < k^3$*,

$$a_{k,h,l} = \frac{3ik}{\nu(k^2 - h) - i\alpha(k^3 - l)} \sum_{k_1+k_2=k} \sum_{h_1+h_2=h} \sum_{1_1+1_2=l} \alpha_{k_1,h_1,l_1}\alpha_{k_2,h_2,l_2}. \quad (3.7)$$

(2) *For $h = k^2$ and $l = k^3$*,

$$a_{k,k^2,k^3} = a_k(0) - \sum_{k \le h < k^2} \sum_{k \le l < k^3} a_{k,h,l}. \quad (3.8)$$

(3) *For $U(k) < h < k^2$ or $V(k) < l < k^3$*,

$$a_{k,h,l} = 0. \quad (3.9)$$

Proof of Proposition 3.1 If (3.2) solves (3.1), then $a_k(t)$ solves the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} a_k(t) + \left(\nu k^2 - \alpha \mathrm{i} k^3\right) a_k(t) - 3\mathrm{i}k \sum_{k_1+k_2=k} a_{k_1}(t) a_{k_2}(t) = 0.$$

The equivalent integral form is given by

$$a_k(t) = \mathrm{e}^{-(\nu k^2 - \alpha i k^3)t} \left[a_{0k} + 3\mathrm{i}k \int_0^t \mathrm{e}^{(\nu k^2 - \alpha i k^3)\tau} \sum_{k_1+k_2=k} a_{k_1}(\tau) a_{k_2}(\tau)\, \mathrm{d}\tau \right]. \quad (3.10)$$

It is easy to show through an inductive process that a_k is of the form (3.4). In addition, for $k \le h < k^2$ and $k \le l < k^3$, the term in (3.5) with fixed $j_1, j_2, \ldots, j_k$ satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k$$

can be expressed as

$$\begin{aligned} & C(\alpha, \nu, k, h, l, j_1, \ldots, j_k) a_{01}^{j_1} a_{02}^{j_2} \cdots a_{0k}^{j_k} \\ &\quad = \frac{3ik}{\nu(k^2 - h) - \mathrm{i}\alpha(k^3 - l)} \sum_{m_1+n_1=j_1} \cdots \sum_{m_k+n_k=j_k} C(\alpha, \nu, k_1, h_1, l_1, m_1, \ldots, m_{k_1}) \\ &\qquad \times C(\alpha, \nu, k_2, h_2, l_2, n_1, \ldots, n_{k_2}) a_{01}^{m_1+n_1} a_{02}^{m_2+n_2} \cdots a_{0k}^{m_k+n_k} \end{aligned} \quad (3.11)$$

where the indices satisfy

$$\begin{aligned} &1 \le k_1 \le k, \qquad 1 \le k_2 \le k, \qquad k_1 + k_2 = k, \\ &k_1 \le h_1 \le k_1^2, \qquad k_2 \le h_2 \le k_2^2, \qquad h_1 + h_2 = h, \\ &k_1 \le l_1 \le k_1^3, \qquad k_2 \le l_2 \le k_2^3, \qquad l_1 + l_2 = l, \end{aligned}$$

$$m_1+n_1=j_1,\qquad m_2+n_2=j_2,\qquad \ldots,\qquad m_k+n_k=j_k$$
$$(m_r=0\quad \text{for } r>k_1\quad \text{and}\quad n_r=0\quad \text{for } r>k_2)$$
$$m_1+2m_2+\cdots+k_1m_{k_1}=k_1,\qquad n_1+2n_2+\cdots+k_2n_{k_2}=k_2.$$

When $h=k^2$ and $l=k^3$,

$$C\big(\alpha,\nu,k,k^2,k^3,j_1,j_2,\ldots,j_k\big)=\begin{cases}1 & \text{for } (j_1,j_2,\ldots,j_k)=(0,0,\ldots,1),\\ -C(\alpha,\nu,k,h,l,j_1,j_2,\ldots,j_k) & \text{otherwise}\end{cases} \tag{3.12}$$

for some $h<k^2$ and $l<k^3$. To illustrate these formulas, we list a_k for $k=1,2,3$,

$$a_1(t)=a_{01}\mathrm{e}^{-(\nu-i\alpha)t},$$
$$a_2(t)=\frac{6\mathrm{i}}{2\nu-6\alpha\mathrm{i}}a_{01}^2\mathrm{e}^{-(2\nu-2\alpha i)t}+\left[a_{02}-\frac{6\mathrm{i}}{2\nu-6\alpha\mathrm{i}}a_{01}^2\right]\mathrm{e}^{-(4\nu-8\mathrm{i}\alpha)t},$$
$$\begin{aligned}a_3(t)={}&\frac{108a_{01}^3}{(2\nu-6\alpha\mathrm{i})(6\nu-24\alpha\mathrm{i})}\mathrm{e}^{(-3\nu+3\alpha\mathrm{i})t}\\&+\left[\frac{18\mathrm{i}a_{01}a_{02}}{4\nu-18\alpha\mathrm{i}}-\frac{108a_{01}^3}{(2\nu-6\alpha\mathrm{i})(4\nu-18\alpha\mathrm{i})}\right]\mathrm{e}^{(-5\nu+9i\alpha)t}\\&+\left[a_{03}-\frac{18\mathrm{i}a_{01}a_{02}}{4\nu-18\alpha\mathrm{i}}+\frac{108a_{01}^3}{(2\nu-6\alpha\mathrm{i})(4\nu-18\alpha\mathrm{i})}-\frac{108a_{01}^3}{(2\nu-6\alpha\mathrm{i})(6\nu-24\alpha\mathrm{i})}\right]\\&\times\mathrm{e}^{(-9\nu+27\alpha i)t}.\end{aligned}$$

□

Proof of Proposition 3.2 Equation (3.7) follows from a simple induction. Equation (3.8) is obtained by setting $t=0$ in (3.4). To show (3.9), we notice that the second summation in (3.7) is over $h_1+h_2=h$ with $k_1\le h_1\le k_1^2$ and $k_2\le h_2\le k_2^2$ while the third summation is over $l_1+l_2=l$ with $k_1\le l_1\le k_1^3$ and $k_2\le l_2\le k_2^3$. Thus,

$$h=h_1+h_2\le k_1^2+k_2^2=k^2-2k_1k_2\le k^2-2(k-1)=U(k),$$
$$l=l_1+l_2\le k_1^3+k_2^3=k^3-3kk_1k_2\le k^3-3k(k-1)=V(k).$$

That means that $a_{k,h,l}=0$ if $U(k)<h<k^2$ and $V(k)<l<k^3$. □

Theorem 3.3 *Consider (3.1) with $\nu>0$. Assume $u_0\in H^s(\mathbf{T})$ with $s>\frac{1}{2}$ can be represented in the form (3.3) with*

$$|a_{0k}|\le 1,\quad k=1,2,\ldots \tag{3.13}$$

If we have the uniform bound

$$\big|C(\alpha,\nu,k,h,l,j_1,\ldots,j_k)\big|\le C_0(\alpha,\nu) \tag{3.14}$$

for all $k\ge 1$, $k\le h<k^2$, $k\le l<k^3$ and $(j_1,j_2,\ldots,j_k)$ satisfying (3.6), then (3.1) has a unique global solution u given by (3.2). In addition, for any $s\ge 0$, there are

$T_0 > 0$ and $\delta > 0$ such that for any $t \geq T_0$,

$$\left\|u(\cdot,t)\right\|_{H^s} < \frac{C(\alpha,\nu,s)}{1-\mathrm{e}^{-\nu t}}\mathrm{e}^{-\delta\nu k t} \tag{3.15}$$

where C is a constant depending on α, ν and s only.

We remark that the assumption in (3.14) can be verified for the case when $a_{01} > 0$ and $a_{02} = a_{03} = \cdots = 0$. We assume that ν and α satisfy $\nu^2 + 9\alpha^2 \geq 36$ and show by induction that

$$\left|C(\alpha,\nu,k,h,l,j_1,\ldots,j_k)\right| \leq 1.$$

Since $a_{02} = a_{03} = \cdots = 0$, these coefficients are nonzero only if $j_1 = k$ and $j_2 = j_3 = \cdots = j_k = 0$. For any $k \leq h < k^2$ and $k \leq l < k^3$, we have, according to (3.11),

$$\begin{aligned}
&\left|C(\alpha,\nu,k,h,l,j_1,\ldots,j_k)\right| \\
&\quad \leq \left|\frac{3\mathrm{i}k}{\nu(k^2-h)-\mathrm{i}\alpha(k^3-l)}\right| \sum_{m_1+n_1=j_1} \left|C(\alpha,\nu,k_1,h_1,l_1,m_1,\ldots,m_{k_1})\right| \\
&\qquad \times \left|C(\alpha,\nu,k_2,h_2,l_2,n_1,\ldots,n_{k_2})\right|.
\end{aligned}$$

For $j_1 = k$, the number of terms in the summation $m_1 + n_1 = j_1$ is at most k. By the inductive assumption,

$$\left|C(\alpha,\nu,k,h,l,j_1,\ldots,j_k)\right| \leq \frac{3k^2}{\sqrt{\nu^2(k^2-h)^2+\alpha^2(k^3-l)^2}}.$$

Applying (3.9), $h \leq U(k) \equiv k^2 - 2k + 2$ and $l \leq V(k) \equiv k^3 - 3k^2 + 3k$ and thus $|C(\alpha,\nu,k,h,l,j_1,\ldots,j_k)| \leq 1$ by taking into account the assumption on ν and α. When $h = k^2$ and $l = k^3$, the boundedness of the coefficient follows from (3.12).

The proof of Theorem 3.3 involves a very classical problem in number theory, namely the number of integer solutions $(j_1, j_2, \ldots, j_k)$ to the equation defined in (3.6) for a given positive integer k. This problem is not as simple as it may look. An upper bound and an asymptotic approximation for the number of nonnegative solutions are given by Hardy and Ramanujan (1918), as stated in the following lemma.

Lemma 3.4 *Let $k > 0$ be an integer and let N_k denote the number of nonnegative solutions to the equation*

$$j_1 + 2j_2 + \cdots + kj_k = k.$$

Then, for some constant C_1,

$$N_k < \frac{C_1}{k}\mathrm{e}^{2\sqrt{2k}}.$$

In addition, N_k has the following asymptotic behavior:

$$N_k \sim \frac{1}{4\sqrt{3}k}\mathrm{e}^{\pi\sqrt{\frac{2k}{3}}}, \quad \text{as } k \to \infty.$$

Proof of Theorem 3.3 Applying (3.13) and (3.14), we obtain the following bound for $a_{k,h,l}$ in (3.4):

$$|a_{k,h,l}| \le C_0(\alpha,\nu)N_k \le \frac{C_2}{k}\mathrm{e}^{2\sqrt{2k}},$$

where $C_2 = C_0C_1$ and we have used Lemma 3.4. Therefore,

$$\begin{aligned}\bigl|a_k(t)\bigr| &\le \sum_{k\le h\le k^2}\sum_{k\le l\le k^3}|a_{k,h,l}|\mathrm{e}^{-\nu ht}\\ &\le C_2\bigl(k^2-1\bigr)\mathrm{e}^{2\sqrt{2}\sqrt{k}}\frac{\mathrm{e}^{-\nu kt}}{1-\mathrm{e}^{-\nu t}}.\end{aligned} \tag{3.16}$$

For any fixed $t>0$, we can choose $K=K(\nu)$ and $0<M=M(\nu)<1$ such that

$$\bigl|a_k(t)\bigr| \le \frac{C_2}{1-\mathrm{e}^{-\nu t}}M^k \quad \text{for } k\ge K.$$

Therefore, u represented by (3.2) converges for any $t>0$. In addition, $u(\cdot,t)\in H^s$ for any $s\ge 0$. To see the exponential decay of $\|u(\cdot,t)\|_{H^s}$ for large time, we choose $T_0=T_0(\nu,s)$ such that for any $t\ge T_0$ and $k\ge 1$

$$\bigl(1+k^2\bigr)^s\bigl|a_k(t)\bigr|^2 \le C_2M_1^k\frac{\mathrm{e}^{-\delta\nu kt}}{1-\mathrm{e}^{-\nu t}},$$

where $M_1>0$ and $\delta>0$ are some constants. This bound then implies (3.15). This completes the proof of Theorem 3.3.

We finally present a direct proof of the fact that (3.2) is global in time for the special case $a_{02}=a_{03}=\cdots=0$. □

Theorem 3.5 *Consider* (3.1) *with* ν *and* α *satisfying* $\nu^2+4\alpha^2\ge 9$. *If*

$$u_0(x)=a_{01}\mathrm{e}^{\mathrm{i}x} \quad \textit{with } |a_{01}|<1,$$

then (3.1) *has a unique global solution, which can be represented by* (3.2). *In addition, for any* $s\ge 0$, $u(\cdot,t)\in H^s$ *for all* $t\ge 0$.

Proof We prove by induction that, for any $t>0$,

$$\bigl|a_k(t)\bigr| \le |a_{01}|^k, \quad k=1,2,\ldots. \tag{3.17}$$

Obviously, $|a_1(t)|\le |a_{01}|$. To prove (3.17) for $k\ge 2$, we recall (3.10), namely

$$a_k(t)=3\mathrm{i}k\mathrm{e}^{-(\nu k^2-\alpha\mathrm{i}k^3)t}\int_0^t \mathrm{e}^{(\nu k^2-\alpha\mathrm{i}k^3)\tau}\sum_{k_1+k_2=k}a_{k_1}(\tau)a_{k_2}(\tau)\,\mathrm{d}\tau.$$

Since $\nu^2+4\alpha^2\ge 9$, we have

$$\bigl|a_2(t)\bigr| \le \left|\frac{3}{2\nu-4\alpha\mathrm{i}}\right||a_{01}|^2\bigl(1-\mathrm{e}^{-(4\nu-8\alpha\mathrm{i})t}\bigr)\le |a_{01}|^2$$

and more generally,

$$\left|a_k(t)\right| \le \left|\frac{3(k-1)}{\nu k - \alpha \mathrm{i} k^2}\right| |a_{01}|^k \left(1 - \mathrm{e}^{-(\nu k^2 - \alpha \mathrm{i} k^3)t}\right) \le |a_{01}|^k.$$

It is then clear that (3.2) converges in H^s with $s \ge 0$ for any $t \ge 0$. This completes the proof of Theorem 3.5. □

4 Conclusion

Complex-valued partial differential equations admit solutions with much richer structures than those of their real-valued counterparts and have recently attracted the interests of many researchers. This work investigates potential finite-time singularities in solutions of complex-valued Burgers and KdV–Burgers equations. One of our main results is the construction of a finite-time singular solution of the complex Burgers equation. An explicit and entire initial condition, periodic in space, is considered and its corresponding solution is shown to blow up in finite time. The initial datum has only one mode but all higher modes come into play because of nonlinearity, and the wave amplitude approaches infinity at a finite time T. We also studied the complex KdV–Burgers equation to pinpoint conditions on the initial Fourier modes so that the corresponding solution remains regular for all time under the interaction among nonlinearity, dispersion, and dissipation.

The study presented in this work is also motivated by the physical applications of these complex-valued equations. These equations arise in various contexts as modeling equations, and the singular behavior of their solutions bears important physical implications. The complex Burgers equation is a governing equation in studying the formation of facets and edges in the limit shapes of interfaces such as crystal surfaces, and the singularity of its solutions corresponds to singular shapes in the interfaces such as cusps (Kenyon and Okounkov 2007). It also serves as a standard quadratic flux model for the real Burgers shock (Liu and Zumbrun 1995). The complex KdV equation with higher-order correction terms models diffusion-controlled directional crystal growth (Kerszberg 1984; Wu and Yuan 2005), and the stationary complex KdV equation models small perturbations about a reference velocity in an irrotational flow (Levi 1994; Levi and Sanielevici 1996). The complex KdV–Burgers equation describes propagation of nonlinear waves and solitons in media with a viscous-type dissipation.

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