



# Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition

Yuelong Xiao<sup>a,b</sup>, Zhouping Xin<sup>b,c</sup>, Jiahong Wu<sup>d,\*</sup>

<sup>a</sup> *Institute for Computational and Applied Mathematics, Xiangtan University, Xiangtan 411105, PR China*

<sup>b</sup> *The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Satin, N.T., Hong Kong*

<sup>c</sup> *Department of Mathematics, The Chinese University of Hong Kong, Satin, N.T., Hong Kong*

<sup>d</sup> *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA*

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## Abstract

This work investigates the solvability, regularity and vanishing viscosity limit of the 3D viscous magnetohydrodynamic system in a class of bounded domains with a slip boundary condition.

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## 1. Introduction

Let  $\Omega \subset \mathbb{R}^3$  be a bounded smooth domain, we consider the initial- and boundary-value problem (IBVP) for the system of viscous MHD equations

$$\partial_t v - \nu \Delta v + (\nabla \times v) \times v + H \times (\nabla \times H) + \nabla p = 0 \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega, \quad (1.2)$$

$$\partial_t H - \mu \Delta H + v \cdot \nabla H - H \cdot \nabla v = 0 \quad \text{in } \Omega, \quad (1.3)$$

$$\nabla \cdot H = 0 \quad \text{in } \Omega \quad (1.4)$$

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\* Corresponding author.

*E-mail addresses:* [xy1@xtu.edu.cn](mailto:xy1@xtu.edu.cn) (Y. Xiao), [zpxin@ims.cuhk.hk](mailto:zpxin@ims.cuhk.hk) (Z. Xin), [jiahong@math.okstate.edu](mailto:jiahong@math.okstate.edu) (J. Wu).

with the following slip without friction boundary conditions

$$v \cdot n = 0, \quad \nabla \times v \cdot \tau = 0 \quad \text{on } \partial\Omega, \quad (1.5)$$

$$H \cdot n = 0, \quad \nabla \times H \cdot \tau = 0 \quad \text{on } \partial\Omega. \quad (1.6)$$

Here  $\nabla \cdot$  and  $\nabla \times$  denote the div and curl operators, and  $n$  the outward normal vector and  $\tau$  any unit tangential vector of  $\partial\Omega$ , respectively. We investigate the solvability, regularity and vanishing viscosity limit of the IBVP (1.1)–(1.6).

The boundary condition (1.5) on the velocity is a special Navier-type slip boundary condition, which allows the fluid to slip at a slip velocity proportional to the shear stress introduced by Navier [20], this type of boundary conditions have been used in many fluid problems (see e.g. [1,3,4,9,15,18,19,21,25,28]). We also observed that the similar boundary condition (1.6) on the magnetic field  $H$  is adaptable to the systems since it ensured the boundary balance of the quantities on the boundary (see Lemma 2.3 and Lemma 2.4 below).

The viscous MHD system in the whole space or with non-slip boundary conditions have been studied extensively and there is a large literature on various topics concerning the MHD system such as the well-posedness in various functional spaces (see e.g. [2,6,8,12–14,24,29,30]). However, very little is known about the MHD system with a slip boundary condition. The solvability of (1.1)–(1.6) is far from being obvious due to the compatibility issues of the nonlinear terms with the slip boundary conditions. To deal with this difficulty, we follow the approach of [31] and formulate the boundary-value problem in a suitable functional setting so that the Stokes operator is well-behaved. In these functional settings, the nonlinear terms naturally fall into desired functional spaces. These facts allow us to establish the existence and regularity of solutions through the Galerkin approximation and appropriate *a priori* bounds.

With this well-posedness theory at our disposal, we pursue the vanishing viscosity limit of (1.1)–(1.6). The issue of vanishing viscosity limits of the Navier-Stokes equations and the viscous MHD equations is classical and of fundamental importance in fluid dynamics and turbulence theory (see e.g. [11,16,17,22,23,26,27,32]). When a non-slip boundary condition is imposed, the vanishing viscosity limit of the MHD equations is not well understood due to the formation of turbulent boundary layer. Mathematically, one difficulty is due to the mismatch between the boundary condition for the viscous MHD system and that for its potential limit, the ideal MHD system. The ideal MHD system is usually equipped with the slip boundary condition, namely

$$\partial_t v + (\nabla \times v) \times v + H \times (\nabla \times H) + \nabla p = 0 \quad \text{in } \Omega, \quad (1.7)$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega, \quad (1.8)$$

$$\partial_t H + v \cdot \nabla H - H \cdot \nabla v = 0 \quad \text{in } \Omega, \quad (1.9)$$

$$\nabla \cdot H = 0 \quad \text{in } \Omega, \quad (1.10)$$

$$v \cdot n = 0, \quad H \cdot n = 0. \quad (1.11)$$

As pointed out in [16], the key in studying the vanishing viscosity limit is to control the vorticity created at the boundary. Thus to obtain a uniform convergence of solutions of (1.1)–(1.6) to that of the ideal problem (1.7)–(1.11), one needs to obtain some uniform estimates on vorticity (see the proof of Proposition 5.1). Our approach here is motivated by the idea introduced in [31] to study the same problem for the Navier–Stokes equations and is based on the following

observations: First, note that (see [28,31]) the boundary conditions in (1.5)–(1.6) are equivalent to

$$v \cdot n = 0, \quad \partial_n v_\tau = 0 \quad \text{on } \partial\Omega, \tag{1.12}$$

$$H \cdot n = 0, \quad \partial_n H_\tau = 0 \quad \text{on } \partial\Omega \tag{1.13}$$

on the flat portions of the boundary  $\partial\Omega$ , where  $v_\tau = v \cdot \tau$  and  $H_\tau = H \cdot \tau$ . Second, and more importantly, on the flat portions of the boundary  $\partial\Omega$ , if  $v$  and  $H$  satisfy (1.12) and (1.13) respectively, so do  $B_1(v, H) = (\nabla \times v) \times v + H \times (\nabla \times H) + \nabla p$  and  $B_2(v, H) = v \cdot \nabla H - H \cdot \nabla v$ , see Propositions 2.5 and 2.6. These facts enable us to obtain high order uniform estimates in the case that the boundary consists of flat portions. It should be noted that this approach encounters great difficulties for general domains as pointed out by [5]. Thus, following [5], we restrict the problem to a cubic domain  $Q = [0, 1]_{per}^2 \times (0, 1)$  with the boundary conditions on two opposite faces  $z = 0$  and  $z = 1$ , and others be assumed periodic, which was called flat boundary case. Then, we are able to show that any regular solution of (1.1)–(1.6) converges to a corresponding solution of the ideal MHD system (1.7)–(1.11) as  $(\nu, \mu) \rightarrow 0$  in the flat boundary case.

The major results are organized into four sections. Section 2 contains several notation and results to be used in the subsequent sections. Section 3 establishes the existence of global weak solutions through the method of Galerkin approximation. Strong solutions are studied in Section 4 for general domains. The vanishing viscosity limit results for the flat boundary case are presented in Section 5.

## 2. Preliminaries

Some results for the Stokes operator are recalled, the functional spaces in which the solutions of (1.1)–(1.6) are sought are provided and the fact that the nonlinear terms of the MHD system are in suitable functional spaces and some calculations on the boundary for the flat boundary case are established.

Throughout the rest of this paper,  $\Omega \subset R^3$  denotes a simply connected domain or the cubic domain  $Q$ , and  $\partial Q = \{(x_1, x_2, x_3); x_3 = 0, 1\} \cap \bar{Q}$ .  $H^s(\Omega)$  with  $s \geq 0$  denotes the standard Sobolev spaces and  $H^{-s}(\Omega)$  with  $s \geq 0$  denotes the dual of  $H_0^s(\Omega)$  (the closure of  $C_0^\infty(\Omega)$  in  $H^s(\Omega)$ ). Correspondingly,  $H^{-s}(Q)$  denotes the dual of the subspace of  $H^s(Q)$  that contains functions periodic in  $x_1$  and  $x_2$  and equal to zero on  $\partial Q$ . For notational convenience,  $\Omega$  and  $Q$  may be omitted when we write these spaces without confusion.

The following lemma (see [10,31]) allows us to control the  $H^s$ -norm of a vector-valued function  $u$  by its  $H^{s-1}$ -norms of  $\nabla \times u$  and  $\nabla \cdot u$ , together with the  $H^{s-\frac{1}{2}}(\partial\Omega)$ -norm of  $u \cdot n$ .

**Lemma 2.1.** *Let  $s \geq 0$  be an integer. Let  $u \in H^s$  be a vector-valued function. Then*

$$\|u\|_s \leq C(\|\nabla \times u\|_{s-1} + \|\nabla \cdot u\|_{s-1} + |n \cdot u|_{s-\frac{1}{2}} + \|u\|_{s-1}). \tag{2.1}$$

A similar result holds if  $u \cdot n$  is replaced by  $u \times n$  (see [7,31]).

**Lemma 2.2.** *Let  $s \geq 0$  be an integer. Let  $u \in H^s(\Omega)$  (an additional assumption  $\int_Q u \cdot \nabla x_3 \, dx = 0$  should be made when  $\Omega = Q$ ). Then*

$$\|u\|_s \leq C(\|\nabla \times u\|_{s-1} + \|\nabla \cdot u\|_{s-1} + |u \times n|_{s-\frac{1}{2}} + \|u\|_{s-1}). \tag{2.2}$$

**Remark.** The assumption  $\int_Q u \cdot \nabla x_3 dx = 0$  is imposed for the cubic domain so that any function  $u$  satisfies this condition and  $\nabla \times u = 0$ ,  $\nabla \cdot u = 0$  in  $Q$  and  $u \times n = 0$  on  $\partial Q$  is identically zero.

Let

$$X = \{u \in L^2; \nabla \cdot u = 0, u \cdot n = 0\}$$

be the Hilbert space with the  $L^2$  inner product, and let

$$V = H^1 \cap X \subset X,$$

$$W = \{u \in V \cap H^2; (\nabla \times u) \times n = 0 \text{ on } \partial\Omega\} \subset X.$$

In addition,  $V^*$  will denote the dual of  $V$ . As special consequences of (2.1) and (2.2), for any  $u \in V$ ,

$$\|u\|_1 \leq C \|\nabla \times u\|.$$

It is easy to check that for any  $u \in W$  and  $v \in V$ ,

$$(-\Delta u, v) = (\nabla \times u, \nabla \times v).$$

Therefore,  $-\Delta$  can be extended to the closure of  $W$  in  $V$ . The extended operator is denoted by  $A$  and its domain by  $D(A)$ . Obviously,

$$W \subseteq D(A) \subset V.$$

The following lemma states that  $A$  is well-behaved in these functional settings.

**Lemma 2.3.** *The Stokes operator  $A = -\Delta$  with  $D(A) = W \subset V$  satisfying*

$$(Au, v) = a(u, v) \equiv \int_{\Omega} \nabla \times u \cdot \nabla \times v dx$$

*is a self-adjoint and positive operator, with its inverse being compact. Consequently, its countable eigenvalues can be listed as*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$$

*and the corresponding eigenvectors  $\{e_j\} \subset W \cap C^\infty$  make an orthogonal complete basis of  $X$ .*

For notational convenience, we still write  $-\Delta$  for  $A$ . Now, we consider the nonlinear terms in these functional settings. For  $v, H \in C^\infty \cap W$ , define

$$B_1(v, H) = (\nabla \times v) \times v + H \times (\nabla \times H) + \nabla p,$$

where  $p$  satisfies

$$\begin{aligned} \Delta p &= \nabla \cdot ((\nabla \times v) \times v + H \times (\nabla \times H)), \\ \nabla p \cdot n &= ((\nabla \times v) \times v + H \times (\nabla \times H)) \cdot n \end{aligned}$$

and

$$B_2(v, H) = v \cdot \nabla H - H \cdot \nabla v.$$

Obviously,  $B_1(v, H) \in X$ . Indeed, we also have  $B_2(v, H) \in X$  that make the balance of the systems on the boundary. Using the boundary condition  $H \cdot n = 0, v \cdot n = 0$  on  $\partial\Omega$ , we have

$$(H \times v) \times n = (H \cdot n)v - (v \cdot n)H = 0 \quad \text{on } \partial\Omega.$$

Since  $X = \nabla \times \{u \in H^1; \nabla \cdot u = 0, n \times u = 0\}$ , which can be verified directly, we have

$$B_2(v, H) \cdot n = \nabla \times (H \times v + \nabla\varphi) \cdot n = 0 \quad \text{on } \partial\Omega,$$

where  $\varphi$  is the solution of the Dirichlet problem

$$\begin{aligned} -\Delta\varphi &= H \times v, \quad \text{in } \Omega, \\ \varphi &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Next, we give some calculation of the nonlinearities on the boundary associated with the boundary conditions in the flat boundary case ( $\Omega = Q$ ) which will be used to get vanishing viscosity limit. It is easy to see that

**Lemma 2.4.** *Let  $u \in C^\infty(\overline{Q})$ . Then the boundary condition  $u \cdot n = 0, (\nabla \times u) \times n = 0$  is equivalent to  $u_3 = 0, \partial_3 u_j = 0, j = 1, 2$  on the boundary.*

It follows from a simple calculation that

$$\begin{aligned} \nabla \times B_1(v, H) &= (v \cdot \nabla(\nabla \times v) - \nabla \times v \cdot \nabla v) \\ &\quad - (H \cdot \nabla(\nabla \times H) - \nabla \times H \cdot \nabla H). \end{aligned}$$

Following the argument in [31], we have the following two propositions:

**Proposition 2.5.** *For  $v, H \in C^\infty(\overline{Q}) \cap W$ ,  $B_1(v, H) \in C^\infty(\overline{Q}) \cap W$ .*

**Proposition 2.6.** *Let  $v, H \in C^\infty(\overline{Q}) \cap W$ . Then  $B_2(v, H) \in C^\infty(\overline{Q}) \cap W$ .*

For the sake of completeness, we present the proof of Proposition 2.6.

**Proof.** Obviously,  $B_2(v, H) \in C^\infty(\overline{Q})$ , and, we have shown  $(B_2(v, H))_3 = B_2(v, H) \cdot n = 0$  on  $\partial Q$ . It remains to show

$$(\nabla \times B_2(v, H))_j = 0, \quad j = 1, 2 \text{ on } \partial Q.$$

As in Lemma 2.4, it follows from the boundary conditions that

$$v_3 = H_3 = 0, \quad \partial_3 v_j = \partial_3 H_j = 0 \quad \text{for } j = 1, 2 \text{ on } \partial Q.$$

Consequently,

$$\partial_{i,3} v_j = 0, \quad \partial_{i,3} H_j = 0 \quad \text{for } i, j = 1, 2 \text{ on } \partial Q.$$

Therefore

$$(\nabla \times B_2(v, H))_1 = -\partial_3(B_2(v, H))_2 \tag{2.3}$$

$$= -(v_1 \partial_{1,3} H_2 + v_2 \partial_{2,3} H_2) + (H_1 \partial_{1,3} v_2 + H_2 \partial_{2,3} v_2) \tag{2.4}$$

$$= 0 \quad \text{on } \partial Q. \tag{2.5}$$

Similarly,

$$(\nabla \times B_2(v, H))_2 = 0 \quad \text{on } \partial Q.$$

This completes the proof of Proposition 2.6.  $\square$

### 3. The weak solutions

This section establishes the global existence of weak solutions to the MHD system (1.1)–(1.6). The approach is the Galerkin approximation following the argument of Constantin and Foias [10]. Here as in the next section, we consider a general smooth bounded simply connected domain in  $R^3$  unless stated otherwise.

**Definition 3.1.**  $(v, H)$  is called a weak solution of (1.1)–(1.6) with the initial data  $(v_0, H_0) \in X$  on the time interval  $[0, T)$  if  $(v, H) \in L^2(0, T; V) \cap C_w([0, T); X)$  satisfies  $(u', H') \in L^1(0, T; V^*)$  and

$$(v', \phi) + v(\omega_v, \nabla \times \phi) + (\omega_v \times v, \phi) + (H \times \omega_H, \phi) = 0,$$

$$(H', \phi) + \mu(\omega_H, \nabla \times \phi) + (v \cdot \nabla H - H \cdot \nabla v, \phi) = 0$$

for all  $\phi \in V$  and for a.e.  $t \in [0, T)$ , and

$$v(0) = v_0, \quad H(0) = H_0,$$

where  $\omega_v = \nabla \times v$  and  $\omega_H = \nabla \times H$ .

The major result of this section is the global existence of a weak solution.

**Theorem 3.2.** Let  $(v_0, H_0) \in X$ . Let  $T > 0$ . Then there exists at least one weak solution  $(u, H)$  of (1.1)–(1.6) on  $[0, T)$  which satisfies the energy inequality

$$\frac{d}{dt} (\|v\|^2 + \|H\|^2) + 2(v\|\nabla \times v\|^2 + \mu\|\nabla \times H\|^2) \leq 0 \tag{3.1}$$

in the sense of distribution.

**Proof.** We start with a sequence of approximate functions  $(v^{(m)}, H^{(m)})$ ,

$$v^{(m)}(t) = \sum_{j=1}^m v_j(t)e_j, \quad H^{(m)}(t) = \sum_{j=1}^m H_j(t)e_j,$$

where  $v_j$  and  $H_j$  for  $j = 1, \dots, m$ , solve the following ordinary differential equations

$$v'_j(t) + \nu\lambda_j v_j(t) + g_j^1(U) = 0, \tag{3.2}$$

$$H'_j(t) + \mu\lambda_j H_j(t) + g_j^2(U) = 0, \tag{3.3}$$

$$v_j(0) = (v_0, e_j), \quad H_j(0) = (H_0, e_j), \tag{3.4}$$

with  $U = (v_1, v_2, \dots, v_m, H_1, H_2, \dots, H_m)$  and

$$g_j^1(U) = (B_1(v^{(m)}, H^{(m)}), e_j),$$

$$g_j^2(U) = (B_2(v^{(m)}, H^{(m)}), e_j).$$

Since  $(g_j^k(U))$  are Lipschitz in  $U$ , (3.2)–(3.4) is locally well posed, say on  $[0, T)$ . Consequently, for any  $t \in [0, T)$ ,  $(v^{(m)}, H^{(m)})$  solves the following system of equations

$$(v^{(m)})' - \nu\Delta v^{(m)} + P_m B_1(v^{(m)}, H^{(m)}) = 0, \tag{3.5}$$

$$(H^{(m)})' - \mu\Delta H^{(m)} + P_m B_2(v^{(m)}, H^{(m)}) = 0, \tag{3.6}$$

$$v^{(m)}(0) = P_m v_0, \quad H^{(m)}(0) = P_m H_0, \tag{3.7}$$

where  $P_m$  denotes the projection of  $X$  onto the space spanned by  $\{e_j\}_1^m$ .

Taking the inner products ((3.5),  $v^{(m)}$ ) and ((3.6),  $H^{(m)}$ ), adding them up, and noting that

$$(P_m B_1(v^{(m)}, H^{(m)}), v^{(m)}) = \int_{\Omega} (H^{(m)} \times (\nabla \times H^{(m)})) \cdot v^{(m)} dx,$$

$$\begin{aligned} (P_m B_2(v^{(m)}, H^{(m)}), v^{(m)}) &= \int_{\Omega} H^{(m)} \cdot \nabla \times (H^{(m)} \times v^{(m)}) dx \\ &= \int_{\Omega} \nabla \times H^{(m)} \cdot (H^{(m)} \times v^{(m)}) dx, \end{aligned}$$

we obtain by simple algebraic identities,

$$\frac{d}{dt} (\|v^{(m)}\|^2 + \|H^{(m)}\|^2) + 2(\nu\|\nabla \times v^{(m)}\|^2 + \mu\|\nabla \times H^{(m)}\|^2) = 0. \tag{3.8}$$

Therefore,

$$(v^{(m)}, H^{(m)}) \text{ is bounded in } L^\infty(0, T; X), \text{ uniformly for } m, \quad (3.9)$$

$$(\nabla \times v^{(m)}, \nabla \times H^{(m)}) \text{ is bounded in } L^2(0, T; V), \text{ uniformly for } m. \quad (3.10)$$

Note that for  $\phi \in V$ , we have

$$|(-\Delta v^{(m)}, \phi)| = |(\nabla \times v^{(m)}, \nabla \times \phi)|.$$

Therefore,

$$\{-\Delta v^{(m)}\} \text{ is bounded in } L^2(0, T; V^*).$$

Similarly,

$$\{-\Delta H^{(m)}\} \text{ is bounded in } L^2(0, T; V^*).$$

For the nonlinear terms, we have, for any  $\phi \in V$ ,

$$\begin{aligned} |(P_m B_1(v^{(m)}, H^{(m)}), \phi)| &= |(B_1(v^{(m)}, H^{(m)}), \phi_m)| \\ &\leq C(\|v^{(m)}\|^{1/2} \|v^{(m)}\|_1^{3/2} + \|v^{(m)}\|^{1/2} \|v^{(m)}\|_1^{3/2}) \|\phi_m\|_1, \end{aligned}$$

where  $\phi_m = P_m \phi$ . Because of the uniform bound for  $\|v^{(m)}\|$  in (3.9) and the bound for  $\|v^{(m)}\|_1$  in (3.10), we obtain

$$\{B_1(v^{(m)}, H^{(m)})\} \text{ is bounded in } L^{4/3}(0, T; V^*).$$

Similarly,

$$\{B_2(v^{(m)}, H^{(m)})\} \text{ is bounded in } L^{4/3}(0, T; V^*).$$

Therefore,

$$((v^{(m)})', (H^{(m)})') \text{ is bounded in } L^{4/3}(0, T; V^*).$$

The rest of the proof is similar to the arguments in Constantin and Foias [10] and thus further details are omitted. This completes the proof of Theorem 3.2.  $\square$

#### 4. The strong solutions

This section studies the local well-posedness of the strong solution of (1.1)–(1.6) corresponding to an initial data  $(v_0, H_0) \in V$  and its higher regularities.

Let  $(v_0, H_0) \in V$  and let  $(v^{(m)}, H^{(m)})$  be the Galerkin approximation constructed in the previous section. To obtain regularity estimates for  $(v^{(m)}, H^{(m)})$ , we set  $\omega_v^{(m)} = \nabla \times v^{(m)}$  and  $\omega_H^{(m)} = \nabla \times H^{(m)}$ , and obtain their equations by taking the curl of (3.5) and (3.6),

$$(\omega_v^{(m)})' - \nu \Delta \omega_v^{(m)} + \Sigma g_j^1 \nabla \times e_j = 0, \tag{4.1}$$

$$(\omega_H^{(m)})' - \mu \Delta \omega_H^{(m)} + \Sigma g_j^2 \nabla \times e_j = 0, \tag{4.2}$$

$$\omega_v^{(m)}(0) = \nabla \times v^{(m)}(0), \quad \omega_H^{(m)}(0) = \nabla \times H^{(m)}(0), \tag{4.3}$$

where we recall  $g_j^1$  satisfies  $\sum_{j=1}^m g_j^1 e_j = P_m B_1(v^{(m)}, H^{(m)})$ . Taking the inner product ((4.1),  $\omega_v^{(m)}$ ) + ((4.2),  $\omega_H^{(m)}$ ), and noting that

$$(\nabla \times e_i, \nabla \times e_j) = \lambda_j (e_i, e_j),$$

we obtain

$$\begin{aligned} & \frac{d}{dt} (\|\omega_v^{(m)}\|^2 + \|\omega_H^{(m)}\|^2) + 2(\nu \|\nabla \times \omega_v^{(m)}\|^2 + \mu \|\nabla \times \omega_H^{(m)}\|^2) \\ & + 2((\nabla \times B_1(v^{(m)}, H^{(m)}), \omega_v^{(m)}) + (\nabla \times B_2(v^{(m)}, H^{(m)}), \omega_H^{(m)})) = 0. \end{aligned} \tag{4.4}$$

Applying the Agmon inequality

$$\|\phi\|_{L^\infty}^2 \leq \|\phi\|_1 \|\phi\|_2, \quad \forall \phi \in H^2,$$

we find

$$\begin{aligned} (\nabla \times B_1(v^{(m)}, H^{(m)}), \omega_v^{(m)}) & \leq C(\|\omega_v^{(m)}\|^{\frac{3}{2}} + \|\omega_H^{(m)}\|^{\frac{3}{2}})(\|\Delta v^{(m)}\|^{\frac{1}{2}} + \|\Delta H^{(m)}\|^{\frac{1}{2}})\|\Delta v^{(m)}\|, \\ (\nabla \times B_2(v^{(m)}, H^{(m)}), \omega_H^{(m)}) & \leq C(\|\omega_v^{(m)}\|^{\frac{3}{2}} + \|\omega_H^{(m)}\|^{\frac{3}{2}})(\|\Delta v^{(m)}\|^{\frac{1}{2}} + \|\Delta H^{(m)}\|^{\frac{1}{2}})\|\Delta H^{(m)}\| \end{aligned}$$

and

$$\frac{d}{dt} (\|\omega_v^{(m)}\|^2 + \|\omega_H^{(m)}\|^2) + \nu \|\Delta v^{(m)}\|^2 + \mu \|\Delta H^{(m)}\|^2 \leq C(\|\omega_v^{(m)}\| + \|\omega_H^{(m)}\|)^6,$$

where  $C$  depends on  $\nu$  and  $\mu$ . Comparing with the ordinary differential equation

$$\frac{d}{dt} y = Cy^3, \tag{4.5}$$

we find that there is time  $T_0 > 0$  such that, for any fixed  $T \in (0, T_0)$

$$\begin{aligned} (v^{(m)}, H^{(m)}) & \text{ is bounded in } L^\infty(0, T; H^1), \\ (v^{(m)}, H^{(m)}) & \text{ is bounded in } L^2(0, T; H^2). \end{aligned}$$

Note that

$$\|P_m(v \times u)\| \leq \|v \times u\| \leq C\|v\|\|u\|_{L^\infty}, \tag{4.6}$$

it follows that

$$\{(v^{(m)})'\}, \{(H^{(m)})'\} \text{ is bounded in } L^2(0, T; L^2). \tag{4.7}$$

The standard compactness results allow us to find a subsequence of  $(v^{(m)}, H^{(m)})$  (still denoted by  $(v^{(m)}, H^{(m)})$ ) and  $(v, H)$  such that

$$\begin{aligned} v^{(m)} &\rightharpoonup v, & H^{(m)} &\rightharpoonup H && \text{in } L^\infty(0, T; H^1) && \text{weak-star,} \\ v^{(m)} &\rightarrow v, & H^{(m)} &\rightarrow H && \text{in } L^2(0, T; H^2) && \text{weakly,} \\ v^{(m)} &\rightarrow v, & H^{(m)} &\rightarrow H && \text{in } L^2(0, T; H^1) && \text{strongly.} \end{aligned}$$

Passing to the limit, we find the weak solution obtained in the previous section may be chosen such that  $(v, H) \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ . From (4.7), we find  $v', H' \in L^2(0, T; L^2)$  and thus  $v, H \in C([0, T]; H^1)$ . We call such solution a strong solution. To show that strong solutions are unique, we consider two strong solutions  $(v_1, H_1)$  and  $(v_2, H_2)$ . Then their difference  $\tilde{v} = v_1 - v_2, \tilde{H} = H_1 - H_2$  satisfies

$$\tilde{v}' - \nu \Delta \tilde{v} + B_1(v_1, H_1) - B_1(v_2, H_2) = 0, \tag{4.8}$$

$$\tilde{H}' - \mu \Delta \tilde{H} + B_2(v_1, H_1) - B_2(v_2, H_2) = 0. \tag{4.9}$$

Taking the inner products  $((5.6), \tilde{v}) + ((5.7), \tilde{H})$ , we find

$$\frac{d}{dt} (\|\tilde{v}\|^2 + \|\tilde{H}\|^2) \leq g(t) (\|\tilde{v}\|^2 + \|\tilde{H}\|^2)$$

on  $[0, T]$  for some positive integrable function  $g(t)$ . Then,  $v_1 = v_2, H_1 = H_2$  follows from  $\tilde{v}(0) = 0, \tilde{H}(0) = 0$  and the Gronwall inequality. From the standard extension method of time evolution, we can conclude

**Theorem 4.1.** *Let  $(v_0, H_0) \in V$ . Then there is  $T^* > 0$  depending on  $\nu, \mu$  and the  $H^1$ -norm of  $(v_0, H_0)$  only such that (1.1)–(1.6) has a unique strong solution  $(v, H)$  on  $[0, T^*)$  satisfying*

$$\begin{aligned} v, H &\in L^2(0, T; W) \cap C([0, T^*); V), \\ v', H' &\in L^2(0, T; X) \end{aligned}$$

for any  $T \in (0, T^*)$ . In addition, the energy equation

$$\begin{aligned} \frac{d}{dt} (\|\omega_v\|^2 + \|\omega_H\|^2) + 2(\nu \|\Delta v\|^2 + \mu \|\Delta H\|^2) \\ + 2(\nabla \times B_1(v, H), \omega_v) + (\nabla \times B_2(v, H), \omega_H) = 0 \end{aligned}$$

holds, where  $\omega_v = \nabla \times v$  and  $\omega_H = \nabla \times H$ .

We emphasize that all results above valid for both  $\Omega$  and  $Q$ . Now, we begin to investigate the vanishing viscosity limit for the flat boundary case, i.e.,  $\Omega = Q$ .

To explore higher regularities of the strong solution, we let  $(v_0, H_0) \in W$  and consider  $\psi_v^{(m)}(t, x) = -\Delta v^{(m)}(t, x), \psi_H^{(m)}(t, x) = -\Delta H^{(m)}(t, x)$ . It follows from (3.5) and (3.6) that

$$(\psi_v^{(m)})' - \nu \Delta \psi_v^{(m)} + \Sigma g_j^1 \lambda_j e_j = 0, \tag{4.10}$$

$$(\psi_H^{(m)})' - \mu \Delta \psi_H^{(m)} + \Sigma g_j^2 \lambda_j e_j = 0, \tag{4.11}$$

$$\psi_v^{(m)}(0) = -P_m(\Delta v_0), \quad \psi_H^{(m)}(0) = -P_m(\Delta H_0). \tag{4.12}$$

According to Propositions 2.5 and 2.6,  $B_k(v^{(m)}, H^{(m)}) \in W$  for  $k = 1, 2$  and we can integrate by parts to obtain, for  $i = 1, 2, \dots, m$ ,

$$\begin{aligned} (-\Delta P_m B_k(v^{(m)}, H^{(m)}), e_i) &= (\Sigma g_j^k \lambda_j e_j, e_i) \\ &= (B_k(v^{(m)}, H^{(m)}), -\Delta e_i) \\ &= (-\Delta B_k(v^{(m)}, H^{(m)}), e_i) \end{aligned}$$

to find

$$-\Delta P_m B_k(v^{(m)}, H^{(m)}) = P_m(-\Delta B_k(v^{(m)}, H^{(m)})). \tag{4.13}$$

Here we have used  $\nabla \times e_i \times n = 0$  and  $\nabla \times B_k(v^{(m)}, H^{(m)}) \times n = 0$  for  $k = 1, 2$ , on the boundary.

Since  $(\nabla \times B_k(v^{(m)}, H^{(m)})) \times n = 0$  on  $\partial Q$ , we integrate by parts and apply Hölder’s inequality to obtain

$$\begin{aligned} (\Delta B_1(v^{(m)}, H^{(m)}), \psi_v^{(m)}) &\leq C(\|v^{(m)}\|_{L^\infty(Q)} \|v^{(m)}\|_2 + \|H^{(m)}\|_{L^\infty(Q)} \|H^{(m)}\|_2) \|\nabla \times \psi_v^{(m)}\| \\ &\quad + C(\|\nabla v^{(m)}\|_{L^4(Q)}^2 + \|\nabla H^{(m)}\|_{L^4(Q)}^2) \|\nabla \times \psi_v^{(m)}\|, \end{aligned}$$

$(\Delta B_2(v^{(m)}, H^{(m)}), \psi_H^{(m)})$  can be similarly bounded. It then follows from the Sobolev embedding  $H^1 \subset L^4$  and Lemma 2.2 that

$$\begin{aligned} \frac{d}{dt} (\|\psi_v^{(m)}\|^2 + \|\psi_H^{(m)}\|^2) &+ 2(\nu \|\nabla \times \psi_v^{(m)}\|^2 + \mu \|\nabla \times \psi_H^{(m)}\|^2) \\ &\leq C(\|\psi_v^{(m)}\|^2 + \|\psi_H^{(m)}\|^2)^2, \end{aligned}$$

where  $C$  depends on  $\nu$  and  $\mu$ . That is,

$$\begin{aligned} (v^{(m)}, H^{(m)}) &\text{ remains bounds in } L^\infty(0, T; H^2), \\ (v^{(m)}, H^{(m)}) &\text{ remains bounds in } L^2(0, T; H^3). \end{aligned}$$

We thus have established the following regularity result.

**Theorem 4.2.** Consider  $\Omega = Q$ . Let  $(v_0, H_0) \in W$ . Then the unique strong solution  $(v, H)$  obtained in Theorem 4.1 belongs to  $C([0, T^*]; W)$ . Moreover,

$$\begin{aligned} (v, H) &\in L^2(0, T; H^3(Q)) \cap C([0, T^*]; W), \\ (v', H') &\in L^2(0, T; V) \end{aligned}$$

for any  $T \in [0, T^*)$  and the energy equation

$$\begin{aligned} \frac{d}{dt} (\|\psi_v\|^2 + \|\psi_H\|^2) + 2(v\|\nabla \times \psi_v\|^2 + \mu\|\nabla \times \psi_H\|^2) \\ + 2((-\Delta B_1(v, H), \psi_v) + (-\Delta B_2(v, H), \psi_H)) = 0 \end{aligned}$$

holds for  $\psi_v = -\Delta v$ ,  $\psi_H = -\Delta H$  and  $t \in [0, T^*)$ .

Similarly, If  $(v_0, H_0) \in W \cap H^3$ , we obtain

$$\begin{aligned} (v^{(m)}, H^{(m)}) & \text{ bounded in } L^\infty(0, T; H^3), \\ (v^{(m)}, H^{(m)}) & \text{ bounded in } L^2(0, T; H^4), \\ ((v^{(m)})', (H^{(m)})') & \text{ bounded in } L^2(0, T; W). \end{aligned}$$

This is obtained by considering the equations for  $(\nabla \times \psi_v^{(m)}, \nabla \times \psi_H^{(m)})$ ,

$$\begin{aligned} \frac{d}{dt} (\|\nabla \times \psi_v^{(m)}\|^2 + \|\nabla \times \psi_H^{(m)}\|^2) + 2(v\|\Delta \psi_v^{(m)}\|^2 + \mu\|\Delta \psi_H^{(m)}\|^2) \\ + ((\nabla \times)^3 B_1(v^{(m)}, H^{(m)}), \nabla \times \psi_v^{(m)}) + ((\nabla \times)^3 B_2(v^{(m)}, H^{(m)}), \nabla \times \psi_H^{(m)}) = 0 \end{aligned}$$

going through a similar process. Thus, we have the following further regularity result.

**Theorem 4.3.** Consider  $\Omega = Q$ . Assume  $(v_0, H_0) \in W \cap H^3(Q)$ . Then the unique strong solution  $(v, H)$  obtained in Theorem 4.1 satisfies

$$\begin{aligned} (v, H) \in L^2(0, T; H^4(Q)) \cap C([0, T^*]; H^3(Q)), \\ (v', H') \in L^2(0, T; W) \end{aligned}$$

and the energy equation

$$\begin{aligned} \frac{d}{dt} (\|\nabla \times \psi_v\|^2 + \|\nabla \times \psi_H\|^2) + 2(v\|\Delta \psi_v\|^2 + \mu\|\Delta \psi_H\|^2) \\ + (\Delta B_1(v, H), \Delta \psi_v) + (\Delta B_2(v, H), \Delta \psi_H) = 0 \end{aligned}$$

for  $\psi_v = -\Delta v$ ,  $\psi_H = -\Delta H$ . Moreover,  $\psi_v, \psi_H$  satisfies

$$(\nabla \times \psi_v) \cdot \tau = 0, \quad (\nabla \times \psi_H) \cdot \tau = 0 \quad \text{on } \partial Q \tag{4.14}$$

for all  $\tau$  tangent to the boundary.

**Remark.** Indeed, we have shown that  $\psi_v, \psi_H$  satisfies

$$\begin{aligned} \partial_t \psi_v - \nu \Delta \psi_v - \Delta(\omega_v \times v + H \times \omega_H) &= 0 \quad \text{in } Q, \\ \partial_t \psi_H - \mu \Delta \psi_H - \Delta(H \cdot \nabla v - v \cdot \nabla H) &= 0 \quad \text{in } Q, \\ \nabla \cdot \psi_v &= 0, \quad \nabla \cdot \psi_H = 0 \quad \text{in } Q, \\ \psi_v \cdot n &= 0, \quad \psi_H \cdot n = 0 \quad \text{on } \partial Q \\ (\nabla \times \psi_v) \cdot \tau &= 0, \quad (\nabla \times \psi_H) \cdot \tau = 0 \quad \text{on } \partial Q \end{aligned} \tag{4.15}$$

for the corresponding solutions.

### 5. The vanishing viscosity limit

This section focuses on the vanishing viscosity limit of the MHD system for the case  $\Omega = Q$ . We start with the following uniform estimate:

**Proposition 5.1.** *Let  $(v_0, H_0) \in W \cap H^3(Q)$ . Then there is a  $T_0 > 0$  depending on  $\|(v_0, H_0)\|_{H^3}$  only such that the strong solution  $v = v(v, \mu)$ ,  $H = H(v, \mu)$  of the MHD system (1.1)–(1.6) with the initial data  $(v_0, H_0)$  obeys the following uniform bound*

$$\|v(\cdot, t)\|_3 + \|H(\cdot, t)\|_3 \leq C \quad \text{for } t \in [0, T_0],$$

where  $C$  is a constant independent of  $\nu$  and  $\mu$ .

**Proof.** According to Theorems 4.1 and 4.3, for any  $\nu, \mu > 0$ , there is  $T^* = T^*(\nu, \mu) > 0$  such that the solution  $v = v(\nu, \mu)$ ,  $H = H(\nu, \mu)$  satisfies

$$v, H \in L^2(0, T; H^4(Q)) \cap C([0, T^*]; H^3(Q)), \tag{5.1}$$

$$\|v\|_1 + \|H\|_1 \rightarrow \infty, \quad \text{as } t \rightarrow T^* \text{ if } T^* < \infty, \tag{5.2}$$

for any  $T < T^*$ , and, for  $\psi_v = -\Delta v(\nu, \mu)$ ,  $\psi_H = -\Delta H(\nu, \mu)$ ,

$$\begin{aligned} \frac{d}{dt} (\|\nabla \times \psi_v\|^2 + \|\nabla \times \psi_H\|^2) + 2(\nu \|\Delta \psi_v\|^2 + \mu \|\Delta \psi_H\|^2) \\ + (\Delta B_1(v, H), \Delta \psi_v) + (\Delta B_2(v, H), \Delta \psi_H) = 0. \end{aligned} \tag{5.3}$$

We claim that  $T^*(\nu, \mu)$  bounded from below for all  $\nu, \mu > 0$ . Due to the boundary condition (4.15), we can integrate by parts to get

$$(\Delta B_1(v, H), \Delta \psi_v) = ((\nabla \times)^3 B_1(v, H), \nabla \times \psi_v)$$

and

$$(\Delta B_2(v, H), \Delta \psi_H) = ((\nabla \times)^3 B_2(v, H), \nabla \times \psi_H).$$

After some calculations, we find

$$(\nabla \times)^3 B_1(v, H) = (v \cdot \nabla) \nabla \times \psi_v - (H \cdot \nabla) \nabla \times \psi_H \\ + \sum_{i,j=1,2,3; i+j=4} F_{i,j}(D^i v, D^j v) - \sum_{i,j=1,2,3; i+j=4} F_{i,j}(D^i H, D^j H)$$

and

$$(\nabla \times)^3 B_2(v, H) = (v \cdot \nabla) \nabla \times \psi_H - (H \cdot \nabla) \nabla \times \psi_v \\ + \sum_{i,j=1,2,3; i+j=4} F_{i,j}(D^i v, D^j H) - \sum_{i,j=1,2,3; i+j=4} F_{i,j}(D^i H, D^j v),$$

where  $F_{i,j}(D^i u, D^j u)$ 's are bilinear forms and  $D^i$ 's are the  $i$ -th order differential operators. Using the basic fact

$$((u \cdot \nabla)v, w) + ((u \cdot \nabla)w, v) = 0$$

valid for any  $H^1$  vectors  $u, v$  and  $w$  with  $\nabla \cdot u = 0$  in  $Q$  and  $u \cdot n = 0$  on  $\partial Q$  and the bound

$$\left\| \sum_{i,j=1,2,3; i+j=4} F_{i,j}(D^i u, D^j v) \right\| \leq C \|u\|_3 \|v\|_3,$$

we get

$$\left| (\Delta B_1(v, H), \Delta \psi_v) + (\Delta B_2(v, H), \Delta \psi_H) \right| \\ \leq C (\|\nabla \times \psi_v\| + \|\nabla \times \psi_H\|)^3$$

after applying Lemma 2.2, where  $C$  is independent of  $\nu$  and  $\mu$ . It then follows from (5.3) that

$$\frac{d}{dt} (\|\nabla \times \psi_v\|^2 + \|\nabla \times \psi_H\|^2) \\ + 2\nu \|\Delta \psi_v\|^2 + 2\mu \|\Delta \psi_v\|^2 \leq C (\|\nabla \times \psi_v\|^2 + \|\nabla \times \psi_H\|^2)^{\frac{3}{2}}.$$

Comparing with the ordinary differential equation

$$y'(t) = Cy(t)^{\frac{3}{2}}, \\ y(0) = \|\nabla \times \psi_v(0)\|^2 + \|\nabla \times \psi_H(0)\|^2$$

and denote by  $T_0$  the blow up time, it follows that

$$T^*(v, \mu) \geq T_0 \quad \text{for all } v, \mu > 0.$$

This completes the proof of Proposition 5.1.  $\square$

**Theorem 5.2.** Let  $(v_0, H_0) \in W \cap H^3(Q)$ . Let  $T_0 > 0$  and let  $v = v(v, \mu)$ ,  $H = H(v, \mu)$  be the corresponding strong solution of the MHD system (1.1)–(1.6) on  $[0, T_0]$ . Then, as  $v, \mu \rightarrow 0$ ,  $(v, H)$  converges to the unique solution  $(v^0, H^0)$  of the ideal MHD system with the same initial data (1.7)–(1.11) in the sense

$$v(v, \mu), H(v, \mu) \rightarrow (v^0, H^0) \quad \text{in } L^q(0, T; H^3(Q)), \tag{5.4}$$

$$v(v, \mu), H(v, \mu) \rightarrow (v^0, H^0) \quad \text{in } C([0, T]; H^2(Q)) \tag{5.5}$$

for any  $1 \leq q < \infty$ .

**Proof.** It follows from Proposition 5.1 that

$$v(v, \mu), H(v, \mu) \quad \text{is uniformly bounded in } C([0, T_0]; H^3(Q)),$$

$$v'(v, \mu), v'(v, \mu) \quad \text{is uniformly bounded in } L^2(0, T_0; W)$$

for all  $v, \mu > 0$ . By the standard compactness result, there is a subsequence  $v_n, \mu_n$  of  $v, \mu$  and vector functions  $v^0, H^0$  such that

$$(v(v_n, \mu_n), H(v_n, \mu_n)) \rightarrow (v^0, H^0) \quad \text{in } L^q(0, T; H^3(Q)),$$

$$(v(v_n, \mu_n), H(v_n, \mu_n)) \rightarrow (v^0, H^0) \quad \text{in } C([0, T]; H^2(Q))$$

for any  $1 \leq q < \infty$ , as  $v_n, \mu_n \rightarrow 0$ . Passing to the limit, we find the limit  $(v^0, H^0)$  solves the following limit equations

$$\partial_t v^0 + (\nabla \times v^0) \times v^0 + H^0 \times (\nabla \times H^0) + \nabla p = 0 \quad \text{in } Q,$$

$$\nabla \cdot v^0 = 0 \quad \text{in } Q,$$

$$\partial_t H^0 + v^0 \cdot \nabla H^0 - H^0 \cdot \nabla v^0 = 0 \quad \text{in } Q,$$

$$\nabla \cdot H^0 = 0 \quad \text{in } Q$$

with the boundary conditions

$$v^0 \cdot n = 0, \quad (\nabla \times v^0) \cdot \tau = 0 \quad \text{on } \partial Q,$$

$$H^0 \cdot n = 0, \quad (\nabla \times H^0) \cdot \tau = 0 \quad \text{on } \partial Q$$

and  $p$  satisfying

$$\Delta p = \nabla \cdot ((\nabla \times v^0) \times v^0) - \nabla \cdot ((\nabla \times H^0) \times H^0),$$

$$\nabla p \cdot n = 0 \quad \text{on } \partial Q.$$

As in the proof of the uniqueness of the strong solutions of the MHD system in the previous section, we can show that  $(v^0, H^0)$  is unique. We then show the convergence of whole sequence.  $\square$

Finally, we present the convergence rate.

**Theorem 5.3.** Let  $(v_0, H_0) \in W \cap H^3(Q)$ . Let  $T_0$  be as in Proposition 5.1. Then

$$\|v(v, \mu) - v^0\|_2^2 + \|H(v, \mu) - H^0\|_2^2 \leq C(T_0)(v + \mu)$$

on the interval  $[0, T_0]$ .

**Proof.** Denote by  $\tilde{v} = v(v, \mu) - v^0$ ,  $\tilde{H} = H(v, \mu) - H^0$ . We find that  $\psi_{\tilde{v}} = -\Delta\tilde{v}$ ,  $\psi_{\tilde{H}} = -\Delta\tilde{H} \in H^1(Q)$  solve

$$\partial_t \psi_{\tilde{v}} - \Delta(B_1(v, H) - B_1(v^0, H^0)) = -v\Delta^2\tilde{v} \quad \text{in } Q, \tag{5.6}$$

$$\partial_t \psi_{\tilde{H}} - \Delta(B_2(v, H) - B_2(v^0, H^0)) = -\mu\Delta^2\tilde{H} \quad \text{in } Q, \tag{5.7}$$

$$\nabla \cdot \tilde{v} = 0, \quad \nabla \cdot \tilde{H} = 0 \quad \text{in } Q, \tag{5.8}$$

$$\tilde{v} \cdot n = 0, \quad \tilde{H} \cdot n = 0 \quad \text{on } \partial Q \tag{5.9}$$

and  $(\nabla \times v) \times n = 0$ ,  $(\nabla \times v^0) \times n = 0$ ,  $(\nabla \times H) \times n = 0$ ,  $(\nabla \times H^0) \times n = 0$ ,  $(\nabla \times)^3 v \times n = 0$ ,  $(\nabla \times)^3 H \times n = 0$  which follows from (4.14). Taking the  $L^2$  inner product of (5.6) with  $\psi_{\tilde{v}}$  and (5.6) with  $\psi_{\tilde{H}}$ , integrating by parts, one can get

$$\begin{aligned} & \frac{d}{dt} (\|\psi_{\tilde{v}}\|^2 + \|\psi_{\tilde{H}}\|^2) - 2(\Delta(B_1(v, H) - B_1(v^0, H^0)), \psi_{\tilde{v}}) \\ & \quad - 2(\Delta(B_2(v, H) - B_2(v^0, H^0)), \psi_{\tilde{H}}) \\ & = v((\nabla \times)^3 \tilde{v}, \nabla \times \psi_{\tilde{v}}) + \mu((\nabla \times)^3 \tilde{H}, \nabla \times \psi_{\tilde{H}}). \end{aligned}$$

As in the proof of Proposition 5.1, we have

$$\begin{aligned} -\Delta(B_1(v, H) - B_1(v^0, H^0)) &= (v \cdot \nabla)\psi_{\tilde{v}} - (H \cdot \nabla)\psi_{\tilde{H}} + \sum_{j=1,2} F_{i,j}(D^i v^0, D^j \tilde{v}) \\ & \quad - \sum_{j=1,2} F_{i,j}(D^i H^0, D^j \tilde{H}), \\ -\Delta(B_2(v, H) - B_2(v^0, H^0)) &= (v \cdot \nabla)\psi_{\tilde{H}} - (H \cdot \nabla)\psi_{\tilde{v}} + \sum_{j=1,2} F_{i,j}(D^i v^0, D^j \tilde{H}) \\ & \quad - \sum_{j=1,2} F_{i,j}(D^i H^0, D^j \tilde{v}), \end{aligned}$$

where the summation is also over index  $i = 1, 2, 3$ . Therefore,

$$\begin{aligned} & |(\Delta(B_1(v, H) - B_1(v^0, H^0)), \psi_{\tilde{v}}) + (\Delta(B_2(v, H) - B_2(v^0, H^0)), \psi_{\tilde{H}})| \\ & \leq C(\|v^0\|_3 + \|H^0\|_3)(\|\psi_{\tilde{v}}\|^2 + \|\psi_{\tilde{H}}\|^2). \end{aligned}$$

Also, we have

$$|((\nabla \times)^3 v, \nabla \times \psi_{\tilde{v}})| \leq C\|(\nabla \times)^3 v\|(\|(\nabla \times)^3 v\| + \|(\nabla \times)^3 v^0\|)$$

and

$$|((\nabla \times)^3 H, \nabla \times \psi_{\tilde{H}})| \leq C\|(\nabla \times)^3 H\|(\|(\nabla \times)^3 H\| + \|(\nabla \times)^3 H^0\|).$$

These estimates are uniform respect to  $v, \mu$  and thus

$$\frac{d}{dt}(\|\psi_{\tilde{v}}\|^2 + \|\psi_{\tilde{H}}\|^2) \leq C(T_0)(\|\psi_{\tilde{v}}\|^2 + \|\psi_{\tilde{H}}\|^2 + v + \mu).$$

Since  $\tilde{v}(0) = 0, \tilde{H}(0) = 0$ , Gronwall's inequality implies

$$\|\psi_{\tilde{v}}\|^2 + \|\psi_{\tilde{H}}\|^2 \leq C(T_0)(v + \mu). \tag{5.10}$$

This completes the proof of Theorem 5.3.  $\square$

### 6. Further remarks

The convergence result of the last section is also valid for each parameter tending to zero individually. In fact, we have the following theorem.

**Theorem 6.1.** *Let  $v_0, H_0 \in W \cap H^3(Q)$ .*

- 1) *Let  $v$  be fixed and let  $T_0(v) > 0$  be the existence time as in Proposition 5.1. Let  $(v(v, \mu), H(v, \mu))$  be the strong solution of the MHD system (1.1)–(1.6) corresponding to  $(v_0, H_0)$ . Then  $(v(v, \mu), H(v, \mu))$  converges to the unique solution  $(v^v, H^v)$  of the limit system*

$$\begin{aligned} \partial_t v^v - v \Delta v^v + (\nabla \times v^v) \times v^v + H^v \times (\nabla \times H^v) + \nabla p &= 0 \quad \text{in } Q, \\ \nabla \cdot v^v &= 0 \quad \text{in } Q, \\ \partial_t H^v + v^v \cdot \nabla H^v - H^v \cdot \nabla v^v &= 0 \quad \text{in } Q, \\ \nabla \cdot H^v &= 0 \quad \text{in } \Omega \end{aligned}$$

with the following boundary conditions

$$\begin{aligned} v^v \cdot n &= 0, & (\nabla \times v^v) \cdot \tau &= 0 \quad \text{on } \partial Q, \\ H^v \cdot n &= 0, & (\nabla \times H^v) \cdot \tau &= 0 \quad \text{on } \partial Q \end{aligned}$$

and the same initial data. The convergence is in the following sense

$$\begin{aligned} v(v, \mu) &\rightarrow v^v, & H(v, \mu) &\rightarrow H^v \quad \text{in } L^q(0, T; H^3(Q)), \\ v(v, \mu) &\rightarrow v^v, & H(v, \mu) &\rightarrow H^v \quad \text{in } C([0, T]; H^2(Q)) \end{aligned}$$

for  $1 \leq q < \infty$ , as  $\mu \rightarrow 0$ .

- 2) *Let  $\mu > 0$  be fixed and let  $T_0(\mu) > 0$  be the existence time as in Proposition 5.1. Let  $(v(v, \mu), H(v, \mu))$  be the strong solution of the MHD system (1.1)–(1.6) corresponding to  $(v_0, H_0)$ . Then  $(v(v, \mu), H(v, \mu))$  converges to the unique solution  $(v^\mu, H^\mu)$  of the limit system*

$$\begin{aligned} \partial_t v^\mu + (\nabla \times v^\mu) \times v^\mu + H^\mu \times (\nabla \times H^\mu) + \nabla p &= 0 \quad \text{in } Q, \\ \nabla \cdot v^\mu &= 0 \quad \text{in } \Omega, \\ \partial_t H^\mu - \mu \Delta H^\mu + v^\mu \cdot \nabla H^\mu - H^\mu \cdot \nabla v^\mu &= 0 \quad \text{in } Q, \\ \nabla \cdot H^\mu &= 0 \quad \text{in } Q \end{aligned}$$

with the following boundary conditions

$$\begin{aligned} v^\mu \cdot n &= 0, & \nabla \times v^\mu \cdot \tau &= 0 \quad \text{on } \partial Q, \\ H^\mu \cdot n &= 0, & \nabla \times H^\mu \cdot \tau &= 0 \quad \text{on } \partial Q \end{aligned}$$

in the sense

$$\begin{aligned} v(v, \mu) &\rightarrow v^\mu, & H(v, \mu) &\rightarrow H^\mu \quad \text{in } L^q(0, T; H^3(Q)), \\ v(v, \mu) &\rightarrow v^\mu, & H(v, \mu) &\rightarrow H^\mu \quad \text{in } C([0, T]; H^2(Q)) \end{aligned}$$

for  $1 \leq q < \infty$ , as  $v \rightarrow 0$ .

**Theorem 6.2.** Let  $(v_0, H_0) \in W \cap H^3(Q)$ . Let  $v$  or  $\mu$  be fixed and let  $T_0(v)$  or  $T_0(\mu)$  be as in the previous theorem. Then we have

$$\|v(v, \mu) - v^v\|_2^2 + \|H(v, \mu) - H^v\|_2^2 \leq C(T_0)\mu$$

for  $t \in [0, T_0(v)]$ , or

$$\|v(v, \mu) - v^\mu\|_2^2 + \|H(v, \mu) - H^\mu\|_2^2 \leq C(T_0)v$$

for  $t \in [0, T_0(v)]$ .

One can further consider the secondary vanishing viscosity limit of  $(v^v, H^v)$  as  $v \rightarrow 0$  or the limit of  $(v^\mu, H^\mu)$  as  $\mu \rightarrow 0$ . In a similar fashion, it can be shown that they both converge to the solution  $(v^0, H^0)$  of the ideal MHD system (1.7)–(1.11).

We remark that it may be possible to consider the partial vanishing viscosity limits for a more general bounded domain. For instance, in the case when  $v$  is fixed and  $\mu \rightarrow 0$ , one can integrate by parts to pass the high derivatives to  $v$  due to the presence of the dissipative term in  $v$  and obtain the following bound

$$\frac{d}{dt} (\|\omega_v\|^2 + \|\omega_H\|^2) + 2(v\|\Delta v\|^2 + \mu\|\Delta H\|^2) \leq C(\|\omega_v\| + \|\omega_H\|)^6$$

where  $C$  is independent of  $\mu$ . Therefore, the partial vanishing viscosity limit as  $\mu \rightarrow 0$  may be considered in  $L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$  without resorting to the special geometry of the domain. We omit further details.

We also remark that the vanishing viscosity limit results are valid for the 2D MHD system with the following slip boundary conditions

$$\begin{aligned} v \cdot n &= 0, & \nabla \times v &= 0 & \text{on } \partial\Omega, \\ H \cdot n &= 0, & \nabla \times H &= 0 & \text{on } \partial\Omega \end{aligned}$$

for the domains  $\Omega$  with flat boundaries. In the case of the 2D Navier–Stokes equations, the vanishing viscosity limit results can be established for general domains (see [5,31]). However, it is not clear if they hold for the 2D MHD equations in a general domain. One reason is that it appears difficult to verify  $\nabla \times B_2 = 0$  on the boundary.

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## References

- [1] Y. Achdou, O. Pironneau, F. Valentin, Effective boundary conditions for laminar flows over periodic rough boundaries, *J. Comput. Phys.* 147 (1998) 187–218.
- [2] R. Agapito, M. Schonbek, Non-uniform decay of MHD equations with and without magnetic diffusion, *Comm. Partial Differential Equations* 32 (2007) 1791–1812.
- [3] E. Bänsch, Finite element discretization of the Navier–Stokes equations with a free capillary surface, *Numer. Math.* 88 (2001) 203–235.
- [4] H. Beirão da Veiga, On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions, *Comm. Pure Appl. Math.* 58 (2005) 552–577.
- [5] H. Beirão da Veiga, F. Crispo, Sharp inviscid limit results under Navier type boundary conditions. An  $L^p$  theory, *J. Math. Fluid Mech.* (2009), in press, doi:10.1007/s00021-009-0295-4.
- [6] D. Biskamp, *Nonlinear Magnetohydrodynamics*, Cambridge University Press, Cambridge, UK, 1993.
- [7] J. Cantarella, D. DeTurck, G. Herman, Vector calculus and the topology of domains in 3-space, *Amer. Math. Monthly* 109 (2002) 409–442.
- [8] Q. Chen, C. Miao, Z. Zhang, The Beale–Kato–Majda criterion for the 3D magneto-hydrodynamics equations, *Comm. Math. Phys.* 275 (2007) 861–872.
- [9] T. Clopeau, A. Mikelić, R. Robert, On the vanishing viscosity limit for the 2D incompressible Navier–Stokes equations with the friction type boundary conditions, *Nonlinearity* 11 (1998) 1625–1636.
- [10] P. Constantin, C. Foias, *Navier–Stokes Equations*, Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 1988.
- [11] P. Constantin, J. Wu, Inviscid limit for vortex patches, *Nonlinearity* 8 (1995) 735–742.
- [12] G. Duvaut, J.L. Lions, Équation en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.* 46 (1972) 241–279.
- [13] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Differential Equations* 213 (2005) 235–254.
- [14] C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Funct. Anal.* 227 (2005) 113–152.
- [15] W. Jäger, A. Mikelić, On the roughness-induced effective boundary conditions for an incompressible viscous flow, *J. Differential Equations* 170 (2001) 96–122.
- [16] T. Kato, Remarks on zero viscosity limit for nonstationary Navier–Stokes flows with boundary, in: *Seminar on Nonlinear Partial Differential Equations*, Berkeley, CA, 1983, in: *Math. Sci. Res. Inst. Publ.*, vol. 2, Springer, New York, 1984, pp. 85–98.
- [17] T. Kato, C. Lai, Nonlinear evolution equations and the Euler flow, *J. Funct. Anal.* 56 (1984) 15–28.

- [18] A. Liakos, Discretization of the Navier–Stokes equations with slip boundary condition, *Numer. Methods Partial Differential Equations* 17 (2001) 26–42.
- [19] P. Mucha, On Navier–Stokes equations with slip boundary conditions in an infinite pipe, *Acta Appl. Math.* 76 (2003) 1–15.
- [20] C.L.M.H. Navier, Sur les lois de l'équilibre et du mouvement des corps élastiques, *Mem. Acad. R. Sci. Inst. France* 6 (1827) 369.
- [21] T. Qian, X.-P. Wang, P. Sheng, Generalized Navier boundary condition for the moving contact line, *Commun. Math. Sci.* 1 (2003) 333–341.
- [22] M. Sammartino, R. Caflisch, Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.* 192 (1998) 433–461;  
M. Sammartino, R. Caflisch, Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. II. Construction of the Navier–Stokes solution, *Comm. Math. Phys.* 192 (1998) 463–491.
- [23] H. Schlichting, K. Gersten, E. Krause, H. Oertel, C. Mayes, *Boundary–Layer Theory*, eighth ed., Springer, 2004.
- [24] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* 36 (1983) 635–664.
- [25] W. Silliman, L.E. Scriven, Separating flow near a static contact line: Slip at a wall and shape of a free surface, *J. Comput. Phys.* 34 (1980) 287–313.
- [26] R. Temam, X. Wang, Boundary layers associated with incompressible Navier–Stokes equations: The noncharacteristic boundary case, *J. Differential Equations* 179 (2002) 647–686.
- [27] Y. Wang, Z. Xin, Zero-viscosity limit of the linearized compressible Navier–Stokes equations with highly oscillatory forces in the half-plane, *SIAM J. Math. Anal.* 37 (2005) 1256–1298.
- [28] J. Watanabe, On incompressible viscous fluid flows with slip boundary conditions, in: *Proceedings of the 6th Japan–China Joint Seminar on Numerical Mathematics*, Tsukuba, 2002, *J. Comput. Appl. Math.* 159 (2003) 161–172.
- [29] J. Wu, Viscous and inviscid magneto-hydrodynamics equations, *J. Anal. Math.* 73 (1997) 251–265.
- [30] J. Wu, Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations* 33 (2008) 285–306.
- [31] Y. Xiao, Z. Xin, On the vanishing viscosity limit for the 3D Navier–Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.* LX (2007) 1027–1055.
- [32] Z. Xin, T. Yanagisawa, Zero-viscosity limit of the linearized Navier–Stokes equations for a compressible viscous fluid in the half-plane, *Comm. Pure Appl. Math.* 52 (1999) 479–541.