1. (40 points) No need to explain how you got the answers on this question.

(a) In $\mathbb{R}^3$, circle those expressions that are always zero (as number or vector), where f is an arbitrary scalar valued function and $\mathbf{F}$ is an arbitrary vectorfield.

$\mathrm{curl}(\nabla f)$ $\quad$ $\mathrm{div}(\nabla f)$ $\quad$ $\nabla(\mathrm{div}\,\mathbf{F})$ $\quad$ $\nabla \cdot \nabla f$ $\quad$ $\nabla \times \nabla f$ $\quad$ $\nabla \cdot \mathrm{curl}\,\mathbf{F}$

(b) Suppose $\nabla^2 f(x, y) = 0$ for all x and y. If $\frac{\partial^2 f}{\partial x^2}(1, 2) = 5$, compute $\frac{\partial^2 f}{\partial y^2}(1, 2)$.

-5

(c) Compute $\mathrm{div}(x^2, y^2, xyz)$

$2x + 2y + xy$

(d) Let $\mathbf{c}(t)$ denote the position of a particle where acceleration $\mathbf{a}(t) = 0$ for all t. If initial velocity $\mathbf{v}(0) = (1, 2, 3)$, and initial position $\mathbf{c}(0) = (0, 0, 0)$. Where is the particle at time $t = 2$.

$(2, 4, 6)$

(e) If D is a region in $\mathbb{R}^3$ of volume 10, compute $\iiint_D 2\,dV$

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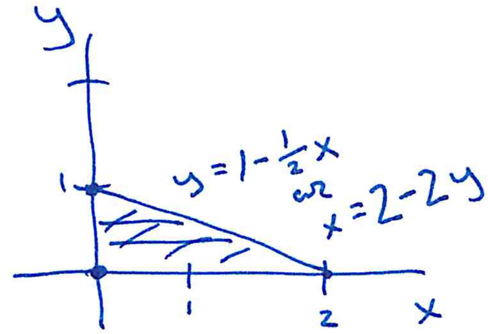
2. (40 points) Let R be the triangle with vertices $(0,0)$, $(0,1)$, and $(2,0)$. Draw a diagram of R and compute

$$\iint_R xy^2\, dA$$



$$\iint_R xy^2\, dA =$$

$$= \int_0^1 \int_0^{2-2y} xy^2\, dx\, dy$$

$$= \int_0^1 \frac{(2-2y)^2}{2} y^2\, dy$$

$$= \int_0^1 (2y^2 - 3y^3 + y^4)\, dy$$

$$= \frac{2}{3} - \frac{3}{4} + \frac{1}{5}$$

3. (40 points) Find minimum/maximum of $\underbrace{x+2y}_{f}$ on the curve $\underbrace{x^2+y^2}_{g} = 5$ using Lagrange multipliers.

$$\nabla f = (1,2)$$
$$\nabla g = (2x, 2y)$$

$$\nabla f = \lambda \nabla g$$

$$(1,2) = \lambda(2x, \cancel{2}y)$$

$$1 = \lambda 2x$$
$$1 = \lambda y \qquad \lambda \neq 0$$

$$y = \frac{1}{\lambda} = 2x$$

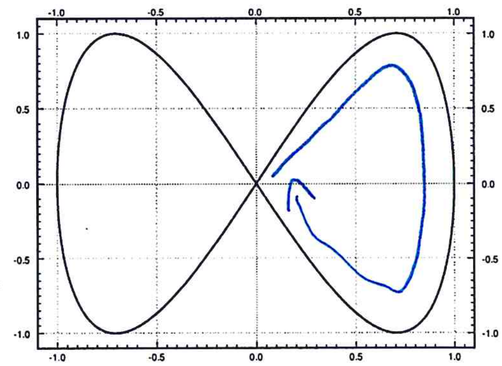
$$5x^2 = x^2 + (2x)^2 = 5$$
$$x^2 = 1 \qquad x = \pm 1$$

at $(1,2)$ at $(-1,-2)$

$f(1,2) = 5$ $f(-1,-2) = -5$

$\uparrow$ max $\uparrow$ min

4. (40 points) Let $\mathbf{r}(t) = \big(\cos(t), \sin(2t)\big)$. The curve given by $\mathbf{r}$ is shown in the plot. Set up the integral to compute the length of **one** of the loops (say the one in $x \geq 0$). No need to evaluate the integral.



$$\int_{-\pi/2}^{\pi/2} \|\vec{r}\,'(t)\|\, dt =$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(-\sin t)^2 + (2\cos(2t))^2}\, dt$$

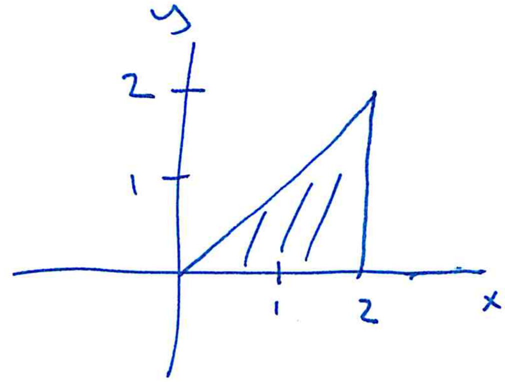
$$= \int_{-\pi/2}^{\pi/2} \sqrt{\sin^2 t + 4\cos^2(2t)}\, dt$$

$\vec{r}(t) = (0,0)$ when $t = \pm \pi/2$

Hint: 1) find the range of t that gives one of the loops 2) Set up the integral.

5. (40 points) Compute

$$\int_0^2 \int_y^2 y\sin(x^3)\,dx\,dy$$



$$= \int_0^2 \int_0^x y \sin(x^3)\,dy\,dx$$

$$= \int_0^2 \frac{x^2}{2} \sin(x^3)\,dx$$

$u = x^3$
$du = 3x^2\,dx$

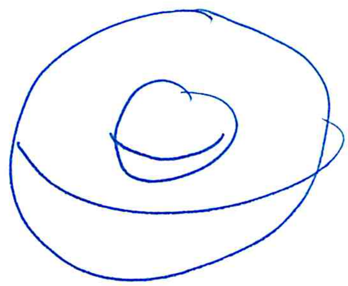
$$= \frac{1}{6} \int_0^8 \sin(u)\,du$$

$$= \frac{1}{6}\left(-\cos(8) + 1\right)$$

6. (10 points (bonus)) If D is the spherical shell given by $1 \leq x^2 + y^2 + z^2 \leq 2$. Compute

$$\iiint_D \frac{\sin(xy)e^{x^4} + x^2 + y^2 + z^2}{x^2 + y^2 + z^2}\, dV$$

(Very little partial credit available (it's a bonus). No points for just guessing the answer without work or explanation. Work on everything else before trying this.)



$\frac{\sin(xy)e^{x^4}}{x^2+y^2+z^2}$ is odd function of x, we are always integrating over symmetric set in x so the integral of that part is zero, so the integral is just

$$\iiint_D \cancel{\frac{\sin(xy)e^{x^4}}{x^2+y^2+z}}^{\,0}\, dV + \iiint_D 1\, dV = \iiint_D 1\, dV = \mathrm{Vol}(D)$$

$$\mathrm{Vol}(D) = \frac{4}{3}\pi(\sqrt{2})^3 - \frac{4}{3}\pi$$