Calculus III Refresher for Vector Calculus

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This class, Vector Calculus, is really the vector calculus that you haven't really gotten to in Calculus III. We will be using the book:

H. M. Schey, *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus* (Fourth Edition)

Let us start with a very quick review of the concepts from Calculus III that we will need and that are not covered in Schey—a crash course if you will. We won't cover nearly everything that you may have seen in Calculus III in this quick overview, just the very basics. We will also go over a couple of things that you may not have seen in Calculus III, but that we will need for this class. You should look back at your Calculus III textbook. If you no longer have that or need another source, there is a wonderful free textbook:

Gregory Hartman, *APEX Calculus*, `http://www.apexcalculus.com`. You can download a PDF online, or buy a very cheap printed copy. Especially Volume 3, that is, chapters 9–14.

1 Vectors

In basic calculus, one deals with $\mathbb{R}$, the real numbers, a one-dimensional space, or the *line*. In vector calculus, we consider the two dimensional cartesian space $\mathbb{R}^2$, the *plane*; three dimensional space $\mathbb{R}^3$; and in general the n-dimensional cartesian space $\mathbb{R}^n$. A *point* in $\mathbb{R}^2, \mathbb{R}^3$, or $\mathbb{R}^n$ is simply a tuple, a 3-tuple, or an n-tuple (respectively) of real numbers. For example, the following are points in $\mathbb{R}^2$

$$(1, -2), \qquad (0, 1), \qquad (-1, 10), \qquad \text{etc.}$$

The following are points in $\mathbb{R}^3$

$$(1, -2, 3), \qquad (0, 0, 1), \qquad (-1, -1, 10), \qquad \text{etc.}$$

Of course, $\mathbb{R}^n$ can be $\mathbb{R}^2$ or $\mathbb{R}^3$, and even $\mathbb{R} = \mathbb{R}^1$, as n can always be 1, 2, or 3. The coordinates used in calculus are x for $\mathbb{R}$, (x, y) for $\mathbb{R}^2$, and (x, y, z) for $\mathbb{R}^3$. In $\mathbb{R}^n$ in general, we run out of letters, and so we use something like subscripts $(x_1, x_2, \ldots, x_n)$. Other letters than x are used just as much. We mostly focus on $\mathbb{R}^3$ (and $\mathbb{R}^2$ to some extent) in this course.

Now that we have points, another object is a *vector*. A vector is an object that describes a *direction* and a *magnitude* (its size or length). It is simply an arrow in space, although it does not really care as to where the arrow starts, it only cares about its direction and its magnitude.

To give vectors names, people often use $\vec{v}$ or $\mathbf{v}$, although mathematicians often just write v and simply remember that v is a vector. On the board I write $\vec{v}$ although the book uses $\mathbf{v}$ (it is difficult to write bold on the board ☺). The book also uses $\hat{\mathbf{v}}$ for unit vectors, that is, vectors of magnitude one. We will write $\hat{v}$.

The best way to think about it is thinking of a moving particle in space. A *point* describes the position of a particle, while a *vector* describes velocity, that is, the direction the object is traveling, and its speed. Forces and displacements are also described by vectors. A vector can say how to go from point A to point B (start at A in this direction and go this far to get to B). Such a vector is written $\overrightarrow{AB}$.

A space $\mathbb{R}^n$ has one special point $O = (0, 0, \ldots, 0)$, the *origin*. We can describe a vector $\vec{v}$ via a point A in space if the vector describes the displacement from O to A, so $\vec{v} = \overrightarrow{OA}$. Then we say $\vec{v}$ is the *position vector* of A. This means that a vector can be described by 3 numbers just like a point. We don't necessarily want to use the same notation as for points, to distinguish a common notation for vectors is

$$\langle a, b, c \rangle,$$

which is the position vector of the point (a, b, c) in $\mathbb{R}^3$. Even though both points and vectors are represented by 3 numbers in $\mathbb{R}^3$, we distinguish them. As far as computations are concerned, they are often just 3 numbers, but they are different things. Just like say temperature, time, or speed are very different things, they are each described by a single number. And we don't want to confuse speed, time, and temperature.

The analogue of the origin is the zero vector $\vec{0}$, for example,

$$\vec{0} = \langle 0, 0, 0 \rangle$$

in $\mathbb{R}^3$. It is the single vector which does not have a well-defined direction, and has a zero magnitude. If you go distance zero, then it doesn't matter in which direction you traveled.

There are a certain number of special vectors called the *standard basis vectors*. In $\mathbb{R}^2$ and $\mathbb{R}^3$ they have special names. In $\mathbb{R}^2$:

$$\hat{\imath} = \langle 1, 0 \rangle, \qquad \hat{\jmath} = \langle 0, 1 \rangle.$$

In $\mathbb{R}^3$:

$$\hat{\imath} = \langle 1, 0, 0 \rangle, \qquad \hat{\jmath} = \langle 0, 1, 0 \rangle, \qquad \hat{k} = \langle 0, 0, 1 \rangle$$

We use hats instead of arrows above the ijk, because these vectors are unit vectors, that is vectors of magnitude 1.

The convenient way to write vectors is using the standard basis. That is, in $\mathbb{R}^2$ write,

$$\langle a, b \rangle = a\hat{\imath} + b\hat{\jmath}, \qquad \text{e.g.} \quad \langle 3, 4 \rangle = 3\hat{\imath} + 4\hat{\jmath}.$$

In $\mathbb{R}^3$ write,

$$\langle a, b, c \rangle = a\hat{\imath} + b\hat{\jmath} + c\hat{k}, \qquad \text{e.g.} \quad \langle 3, 4, -2 \rangle = 3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}.$$

We also allow arithmetic with vectors. First, scalar multiplication. Real numbers are called *scalars* when vectors are around, because they are used to "scale" the vectors. If α is a scalar and $\vec{v}$ is a vector then the product $\alpha\vec{v}$ is the vector with the same direction as $\vec{v}$ (as long as $\alpha \geq 0$) and magnitude multiplied by α. If $\alpha < 0$, then the direction is reversed and the magnitude is multiplied by $|\alpha|$. It turns out that

$$\alpha(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) = \alpha a\hat{\imath} + \alpha b\hat{\jmath} + \alpha c\hat{k} \qquad \text{e.g.} \quad 2(3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}) = 6\hat{\imath} + 8\hat{\jmath} - 4\hat{k}.$$

We can also add vectors. Vector addition is defined by using the displacement interpretation of vectors. If $\vec{v}$ and $\vec{w}$ are vectors then $\vec{v} + \vec{w}$ is the vector where we travel along $\vec{v}$ first and then along $\vec{w}$. It turns out that

$$(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) + (d\hat{\imath} + e\hat{\jmath} + f\hat{k}) = (a + d)\hat{\imath} + (b + e)\hat{\jmath} + (c + f)\hat{k}$$

e.g.
$$(\hat{i} + 2\hat{j} + 3\hat{k}) + (5\hat{i} + \hat{j} - 3\hat{k}) = 6\hat{i} + 2\hat{j} + 0\hat{k} = 6\hat{i} + 2\hat{j}.$$

We write the magnitude of $\vec{v}$ as $|\vec{v}|$. The following formulas compute the magnitude of a vector. In $\mathbb{R}^2$:
$$|a\hat{i} + b\hat{j}| = \sqrt{a^2 + b^2}$$

and in $\mathbb{R}^3$:
$$|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$$

Sometimes when given a vector $\vec{r}$, its magnitude is written simply as r, rather than $|\vec{r}|$. You may have seen $\|\vec{v}\|$ for magnitude. It is the same thing. We will use single bars in this course to match the book.

The direction of $\vec{v}$ written $\hat{v}$ is then the vector
$$\hat{v} = \frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}.$$

Although we'll try to state explicitly that $\hat{v}$ is the direction of $\vec{v}$. Notice again that we put a hat instead of an arrow on unit vectors. It is relatively common to use $\hat{v}$ for a unit vector, even if there was no $\vec{v}$ to begin with. We did this with $\hat{i}$, $\hat{j}$, and $\hat{k}$.

All of these notions are generalized to $\mathbb{R}^n$ in the obvious manner. Higher number of dimensions do occur naturally. For example, if t is time, then time-space can have the coordinates (x, y, z, t), that is $\mathbb{R}^4$. Similarly, the space of all configurations of two particles in 3-space is really $\mathbb{R}^6$, that is $(x_1, y_1, z_1, x_2, y_2, z_2)$, where (x_1, y_1, z_1), is the position of the first particle and (x_2, y_2, z_2) is the position of the second. Similarly to the space of possible position–velocity configurations (phase space) of a single particle has 6 dimensions (3 dimensions for position and 3 for velocity). And if we are modeling liquid by pretending it is a 1000 particles (liquid, after all, is a whole bunch of particles), then the phase space has dimensionn 6000.

2 Products of vectors

We saw one product, that is product of a scalar and a vector:
$$\alpha\vec{v}.$$

Another type of product is the so-called *dot product*
$$(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (d\hat{i} + e\hat{j} + f\hat{k}) = ad + be + cf.$$

E.g.
$$(3\hat{i} + \hat{j} - 2\hat{k}) \cdot (-2\hat{i} + 5\hat{j} + \hat{k}) = -6 + 5 - 2 = -3.$$

This product is easy to generalize to any number of dimensions in the obvious way. Notice that the result of this product is a scalar and not a vector. For this reason it is sometimes called the scalar product. The dot product can compute the magnitude of a vector:
$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}.$$

Geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$, this product is
$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. So the dot product can be used to compute the angle. It doesn't matter if you think of the angle between $\vec{v}$ and $\vec{w}$ or vice-versa, as we are taking the cosine here. Also, there are two ways you could define the angle depending which direction you start in, but because of the cosine you get the same dot product. Two vectors are orthogonal (at right angle, perpendicular) if their dot product is zero.

The dot product is bilinear (if something is called a product, usually people want it to be bilinear):

$$(\alpha\vec{v} + \beta\vec{w}) \cdot \vec{u} = \alpha(\vec{v} \cdot \vec{u}) + \beta(\vec{w} \cdot \vec{u})$$

and

$$\vec{u} \cdot (\alpha\vec{v} + \beta\vec{w}) = \alpha(\vec{u} \cdot \vec{v}) + \beta(\vec{u} \cdot \vec{w}).$$

It is also commutative (not all products are commutative):

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}.$$

Another type of product, which really only exists in $\mathbb{R}^3$, is the *cross product*, sometimes called the *vector product*. This product results in a vector. Geometrically

$$\vec{v} \times \vec{w} = |\vec{v}||\vec{w}|(\sin\theta)\hat{n}.$$

Where θ is the angle going from $\vec{v}$ to $\vec{w}$ in the plane spanned by them (now the order matters), and $\hat{n}$ is the normal vector to that plane oriented according to the right hand rule. The orientation can be figured out from the formula

$$\hat{\imath} \times \hat{\jmath} = \hat{k}.$$

That is, $\hat{k}$ is the normal vector to the xy-plane using the right hand rule.

There are a bunch of ways to compute the cross product, though perhaps the easiest to remember is using algebra. First, the cross product is bilinear:

$$(\alpha\vec{v} + \beta\vec{w}) \times \vec{u} = \alpha(\vec{v} \times \vec{u}) + \beta(\vec{w} \times \vec{u})$$

and

$$\vec{u} \times (\alpha\vec{v} + \beta\vec{w}) = \alpha(\vec{u} \times \vec{v}) + \beta(\vec{u} \times \vec{w}).$$

It is anti-commutative:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}.$$

From this we find

$$\vec{v} \times \vec{v} = \vec{0}.$$

To compute the product we can use the identities:

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}.$$

All three identities list $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ in the same order. If you go in the opposite order you get a minus sign:

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}.$$

Example:

$$(3\hat{\imath} + \hat{k}) \times (\hat{\jmath} + 2\hat{k}) = 3\hat{\imath} \times \hat{\jmath} + (3\cdot 2)\hat{\imath} \times \hat{k} + \hat{k} \times \hat{\jmath} + 2\hat{k} \times \hat{k} = 3\hat{k} + 6(-\hat{\jmath}) + (-\hat{\imath}) + 2\vec{0} = -\hat{\imath} - 6\hat{\jmath} + 3\hat{k}.$$

A very important property of the cross product is that it is orthogonal to both of the vectors. In terms of the dot product:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0, \qquad \vec{w} \cdot (\vec{v} \times \vec{w}) = 0.$$

It is a useful easy to compute way to find the orthogonal vector to a plane. For example, as we computed above $(3\hat{\imath} + \hat{k}) \times (\hat{\jmath} + 2\hat{k}) = -\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$, so $-\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$ is orthogonal (at right angle) to $3\hat{\imath} + \hat{k}$ and also to $\hat{\jmath} + 2\hat{k}$.

A common mnemonic for computing the cross product is using the determinant formula:

$$(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) \times (d\hat{\imath} + e\hat{\jmath} + f\hat{k}) = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ d & e & f \end{bmatrix} = (bf - ce)\hat{\imath} - (af - cd)\hat{\jmath} + (ae - bd)\hat{k}.$$

(Do not forget the minus sign on the $\hat{\jmath}$.)

Perhaps the correct way to define cross product, and also a way to figure out many of its properties from the properties of the dot product and the determinant is the *triple product*. Let $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$, $\vec{v} = v_1\hat{\imath} + c_2\hat{\jmath} + v_3\hat{k}$, and $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$. Then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

3 Functions and partial derivatives

A function is simply an assignment of inputs to some output. A real-valued[1] function of 3 real variables can be written as

$$w = f(x, y, z).$$

That is, given numbers x, y, and z, the function f returns the number $f(x, y, z)$. For example, the temperature in a room where x, y, z are some coordinates on the room, then the temperature $T(x, y, z)$ is a function. Sometimes this is called a "scalar function" or a "scalar field."

If we keep y and z fixed, then the assignment that takes x to $f(x, y, z)$ is a function of one variable. Its derivative is the so-called *partial derivative* with respect to x, written $\frac{\partial f}{\partial x}$. That is

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similarly we define $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ and so on. These partial derivatives are again functions of $\mathbb{R}^3$. We will not use the notation f_x in this course for the partial derivative as it may get confusing; we reserve the subscript for another concept.

To compute partials we simply consider all other variables constant. For example, if $f(x, y, z) = x^2yz + xy + z$ then

$$\frac{\partial f}{\partial x}(x, y, z) = 2xyz + y, \qquad \frac{\partial f}{\partial y}(x, y, z) = x^2z + x, \qquad \text{and} \qquad \frac{\partial f}{\partial z}(x, y, z) = x^2y + 1.$$

As these are functions we may take the derivative again. Let us show a couple of examples,

$$\frac{\partial^2 f}{\partial x \partial y}(x, y, z) = 2xz + 1, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(x, y, z) = 2yz.$$

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[1] By *real* we just mean the real numbers, as opposed to complex numbers which we will not worry about.

The generalization to n variables is similar. E.g., if $f(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3 + x_2 x_4$ then

$$\frac{\partial f}{\partial x_1}(x_1, x_2, x_3, x_4) = 2x_1 x_2 x_3, \qquad \text{or} \qquad \frac{\partial f}{\partial x_2}(x_1, x_2, x_3, x_4) = x_1^2 x_3 + x_4$$

4 Multiple integrals

In the plane the area of a small rectangle with sides Δx and Δy is $\Delta A = \Delta x \Delta y$. If we build a box of height c above this small rectangle, the volume of the box is $c\Delta x \Delta y = c \Delta A$.

Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$, and suppose f is a function of two variables defined for values in R. That is, R is all the points (x, y) such that $a \leq x \leq b$ and $c \leq y \leq d$, and for every (x, y) in R we have a value $f(x, y)$. We divide R into a bunch of rectangles of area $\Delta A = \Delta x \Delta y$. In each of these rectangles we pick a point (x_j, y_j) and then

$$f(x_j, y_j) \Delta A$$

is a reasonable approximation for the volume under the graph of f above the little rectangle. We sum all these approximations

$$\sum_j f(x_j, y_j) \Delta A,$$

which is a reasonable approximation for the volume under the graph of f above the rectangle R. The double integral of f over R is the limit of this expression as Δx and Δy go to zero, and therefore as ΔA goes to zero:

$$\iint_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum_j f(x_j, y_j) \Delta A,$$

For a reasonable function (e.g. continuous), this limit exists. The sum can be done column-wise or row-wise, in which case we have a double sum, and taking the limits we find that

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Notice the units of this quantity. It is the units of f times the units of x times the units of y. So for example if all three are meters, then the unit of dA is m^2 and the unit of $f(x, y)\, dA$ and therefore of $\iint_R f(x, y)\, dA$ is m^3 or volume.

For example,

$$\iint_R dA = A(R) = (b - a)(d - c).$$

Here $A(R)$ is the area of of R. Another example: Let $R = [0, 1] \times [0, 1]$

$$\iint_R xy\, dA = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{1}{2} y\, dy = \frac{1}{4}.$$

Similarly in $\mathbb{R}^3$, a little box of sides Δx, Δy, Δz is of volume $\Delta V = \Delta x \Delta y \Delta z$. Let $B = [a, b] \times [c, d] \times [e, f]$. We follow the same procedure above to find for a function $f(x, y, z)$

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

All the other orderings of dx, dy, and dz also work. Notice the units. If all dimensions are in meters, then $\iiint_B f(x,y)\,dV$ is in units m^4, or 4-dimensional volume.

Sometimes we might just write

$$\iiint_B f\,dV$$

for simplicity.

We could generalize this further. Given $f(x_1,\ldots,x_n)$ on $\mathbb{R}^n$, we start with the n dimensional volume element dV_n or just dV, we integrate f, and obtain $n+1$ dimensional volume "under the graph." Mathematicians sometimes do not make distinction for $n=1$ and $n=2$, and simply call everything "volume,' so 1-dimensional volume is simply length, 2-dimensional volume is area, etc. Mathematicians also often do not use $\iint$ and $\iiint$, and so on, and simply use $\int$, as we all know in what dimension we are integrating based on the nature of the element dV or dA or whatnot.

An integral over a region that is not a rectangle is achieved by setting the function to zero outside this region and then integrating over a large rectangle. It can also be achieved by using iterated integrals and integrating over the changing the limits as appropriate. For example, given a triangle T with vertices at $(0,0)$, $(2,0)$, and $(2,1)$, let us compute the integral of x^2y T. We can describe T as either $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$, or as $0 \leq y \leq 1$, $2y \leq x \leq 2$. Let us compute

$$\iint_T x^2 y\,dA = \int_0^2 \int_0^{x/2} x^2 y\,dy\,dx = \int_0^2 x^2 \frac{(x/2)^2}{2}\,dx = \int_0^2 \frac{x^4}{8}\,dx = \frac{2^5}{8 \cdot 5} = \frac{4}{5},$$

or

$$\iint_T x^2 y\,dA = \int_0^1 \int_{2y}^2 x^2 y\,dx\,dy = \int_0^1 \left(\frac{2^3 y}{3} - \frac{(2y)^3 y}{3} \right) dy = \int_0^1 \frac{8y - 8y^4}{3}\,dy = \frac{4}{5}.$$

Not surprisingly we got the same answer of course, since it is the same region.

The integral is additive, that is, if B is the disjoint union of B_1 and B_2, then

$$\iint_B f\,dA = \iint_{B_1} f\,dA + \iint_{B_2} f\,dA.$$

Also the integral is linear, that is if α, β are real numbers and f and g are functions then

$$\iint_B \alpha f + \beta g\,dA = \alpha \iint_B f\,dA + \beta \iint_B g\,dA.$$

Understanding the above development of the integral is important in applications for recognizing quantities computed by integration. In fact, because calculus is so powerful, sometimes the approximation goes the other way. Instead of the sum being an approximation for the integral, the integral can be computed to approximate a sum. In some sense this is always the case, as our world is really composed of tiny bits rather than continuous unbroken things. But adding up a finite but large number of bits tends to be far harder to do than "adding up infinitely many," that is, integrating. A recurring theme in mathematics is that by making a problem more complicated (in just the right way), we make an impossible-to-solve problem into a tractable, solvable problem.