Chapter 10

Multivariable integral

10.1 Riemann integral over rectangles

Note: ??? lectures

As in chapter chapter 5, we define the Riemann integral using the Darboux upper and lower integrals. The ideas in this section are very similar to integration in one dimension. The complication is mostly notational. The differences between one and several dimensions will grow more pronounced in the sections following.

10.1.1 Rectangles and partitions

Definition 10.1.1. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be such that $a_k \leq b_k$ for all k. A set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is called a *closed rectangle*. In this setting it is sometimes useful to allow $a_k = b_k$, in which case we think of $[a_k, b_k] = \{a_k\}$ as usual. If $a_k < b_k$ for all k, then a set of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ is called an *open rectangle*.

For an open or closed rectangle $R := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ or $R := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$, we define the *n-dimensional volume* by

$$V(R) := (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

A *partition* P of the closed rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is a finite set of partitions $P_1, P_2, \ldots, P_n$ of the intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$. We will write $P = (P_1, P_2, \ldots, P_n)$. That is, for every k there is an integer ℓ_k and the finite set of numbers $P_k = \{x_{k,0}, x_{k,1}, x_{k,2}, \ldots, x_{k,\ell_k}\}$ such that

$$a_k = x_{k,0} < x_{k,1} < x_{k,2} < \cdots < x_{k,\ell_k - 1} < x_{k,\ell_k} = b_k.$$

Picking a set of n integers $j_1, j_2, \ldots, j_n$ where $j_k \in \{1, 2, \ldots, \ell_k\}$ we get the *subrectangle*

$$[x_{1,j_1-1}\ ,\ x_{1,j_1}] \times [x_{2,j_2-1}\ ,\ x_{2,j_2}] \times \cdots \times [x_{n,j_n-1}\ ,\ x_{n,j_n}].$$

For simplicity, we order the subrectangles somehow and we say $\{R_1, R_2, \ldots, R_N\}$ are the subrectangles corresponding to the partition P of R. Or more simply, we say they are the subrectangles of P. In other words we subdivide the original rectangle into many smaller subrectangles. It is not difficult to see that these subrectangles cover our original R, and their volume sums to that of R. That is

$$R = \bigcup_{j=1}^{N} R_j, \qquad \text{and} \qquad V(R) = \sum_{j=1}^{N} V(R_j).$$

When

$$R_k = [x_{1,j_1-1}\ ,\ x_{1,j_1}] \times [x_{2,j_2-1}\ ,\ x_{2,j_2}] \times \cdots \times [x_{n,j_n-1}\ ,\ x_{n,j_n}]$$

then

$$V(R_k) = \Delta x_{1,j_1} \Delta x_{2,j_2} \cdots \Delta x_{n,j_n} = (x_{1,j_1} - x_{1,j_1-1})(x_{2,j_2} - x_{2,j_2-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $f\colon R \to \mathbb{R}$ be a bounded function. Let P be a partition of $[a,b]$ and suppose that there are N subrectangles. Let R_i be a subrectangle of P. Define

$$\begin{aligned}
m_i &:= \inf\{f(x) : x \in R_i\}, \\
M_i &:= \sup\{f(x) : x \in R_i\}, \\
L(P,f) &:= \sum_{i=1}^{N} m_i V(R_i), \\
U(P,f) &:= \sum_{i=1}^{N} M_i V(R_i).
\end{aligned}$$

We call $L(P,f)$ the *lower Darboux sum* and $U(P,f)$ the *upper Darboux sum*.

The indexing in the definition may be complicated, fortunately we generally do not need to go back directly to the definition often. We start proving facts about the Darboux sums analogous to the one-variable results.

Proposition 10.1.2. *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f\colon R \to \mathbb{R}$ is a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. For any partition P of R we have*

$$mV(R) \leq L(P,f) \leq U(P,f) \leq MV(R).$$

Proof. Let P be a partition. Then note that $m \leq m_i$ for all i and $M_i \leq M$ for all i. Also $m_i \leq M_i$ for all i. Finally $\sum_{i=1}^{N} V(R_i) = V(R)$. Therefore,

$$\begin{aligned}
mV(R) = m\left(\sum_{i=1}^{N} V(R_i)\right) = \sum_{i=1}^{N} mV(R_i) \leq \sum_{i=1}^{N} m_i V(R_i) \leq \\
\leq \sum_{i=1}^{N} M_i V(R_i) \leq \sum_{i=1}^{N} MV(R_i) = M\left(\sum_{i=1}^{N} V(R_i)\right) = MV(R). \quad \square
\end{aligned}$$

10.1.2 Upper and lower integrals

By Proposition 10.1.2 the set of upper and lower Darboux sums are bounded sets and we can take their infima and suprema. As before, we now make the following definition.

Definition 10.1.3. If $f \colon R \to \mathbb{R}$ is a bounded function on a closed rectangle $R \subset \mathbb{R}^n$. Define

$$\underline{\int_R} f := \sup\{L(P,f) : P \text{ a partition of } R\}, \qquad \overline{\int_R} f := \inf\{U(P,f) : P \text{ a partition of } R\}.$$

We call $\underline{\int}$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral*.

As in one dimension we have refinements of partitions.

Definition 10.1.4. Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $P = (P_1, P_2, \ldots, P_n)$ and $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n)$ be partitions of R. We say $\widetilde{P}$ a *refinement* of P if as sets $P_k \subset \widetilde{P}_k$ for all $k = 1, 2, \ldots, n$.

It is not difficult to see that if $\widetilde{P}$ is a refinement of P, then subrectangles of P are unions of subrectangles of $\widetilde{P}$. Simply put, in a refinement we took the subrectangles of P and we cut them into smaller subrectangles.

Proposition 10.1.5. *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle, P is a partition of R and $\widetilde{P}$ is a refinement of P. If $f \colon R \to \mathbb{R}$ be a bounded function, then*

$$L(P,f) \leq L(\widetilde{P},f) \qquad \text{and} \qquad U(\widetilde{P},f) \leq U(P,f).$$

Proof. Let $R_1, R_2, \ldots, R_N$ be the subrectangles of P and $\widetilde{R}_1, \widetilde{R}_2, \ldots, \widetilde{R}_M$ be the subrectangles of $\widetilde{R}$. Let I_k be the set of indices j such that $\widetilde{R}_j \subset R_k$. We notice that

$$R_k = \bigcup_{j \in I_k} \widetilde{R}_j, \qquad V(R_k) = \sum_{j \in I_k} V(\widetilde{R}_j).$$

Let $m_j := \inf\{f(x) : x \in R_j\}$, and $\widetilde{m}_j := \inf\{f(x) :\in \widetilde{R}_j\}$ as usual. Notice also that if $j \in I_k$, then $m_k \leq \widetilde{m}_j$. Then

$$L(P,f) = \sum_{k=1}^{N} m_k V(R_k) = \sum_{k=1}^{N} \sum_{j \in I_k} m_k V(\widetilde{R}_j) \leq \sum_{k=1}^{N} \sum_{j \in I_k} \widetilde{m}_j V(\widetilde{R}_j) = \sum_{j=1}^{M} \widetilde{m}_j V(\widetilde{R}_j) = L(\widetilde{P},f). \qquad \square$$

The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

Proposition 10.1.6. *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f \colon R \to \mathbb{R}$ a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. Then*

$$mV(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq MV(R). \tag{10.1}$$

Proof. For any partition P, via Proposition 10.1.2

$$mV(R) \leq L(P,f) \leq U(P,f) \leq MV(R).$$

By taking suprema of $L(P,f)$ and infima of $U(P,f)$ over all P we obtain the first and the last inequality.

The key of course is the middle inequality in (10.1). Let $P = (P_1, P_2, \ldots, P_n)$ and $Q = (Q_1, Q_2, \ldots, Q_n)$ be partitions of R. Define $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n)$ by letting $\widetilde{P}_k = P_k \cup Q_k$. Then $\widetilde{P}$ is a partition of R as can easily be checked, and $\widetilde{P}$ is a refinement of P and a refinement of Q. By Proposition 10.1.5, $L(P,f) \leq L(\widetilde{P},f)$ and $U(\widetilde{P},f) \leq U(Q,f)$. Therefore,

$$L(P,f) \leq L(\widetilde{P},f) \leq U(\widetilde{P},f) \leq U(Q,f).$$

In other words, for two arbitrary partitions P and Q we have $L(P,f) \leq U(Q,f)$. Via Proposition **??** we obtain

$$\sup\{L(P,f) : P \text{ a partition of } R\} \leq \inf\{U(P,f) : P \text{ a partition of } R\}.$$

In other words $\underline{\int_R} f \leq \overline{\int_R} f$. □

10.1.3 The Riemann integral

We now have all we need to define the Riemann integral in n-dimensions over rectangles. Again, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Definition 10.1.7. Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $f : R \to \mathbb{R}$ be a bounded function such that

$$\underline{\int_R} f(x)\, dx = \overline{\int_R} f(x)\, dx.$$

Then f is said to be *Riemann integrable*. The set of Riemann integrable functions on R is denoted by $\mathscr{R}(R)$. When $f \in \mathscr{R}(R)$ we define the *Riemann integral*

$$\int_R f := \underline{\int_R} f = \overline{\int_R} f.$$

When the variable $x \in \mathbb{R}^n$ needs to be emphasized we write

$$\int_R f(x)\, dx, \qquad \int_R f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n, \qquad \text{or} \qquad \int_R f(x)\, dV.$$

If $R \subset \mathbb{R}^2$, then often instead of volume we say area, and hence write

$$\int_R f(x)\, dA.$$

Proposition 10.1.6 implies immediately the following proposition.

Proposition 10.1.8. *Let $f\colon R \to \mathbb{R}$ be a Riemann integrable function on a closed rectangle $R \subset \mathbb{R}^n$. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in R$. Then*

$$mV(R) \leq \int_R f \leq MV(R).$$

Example 10.1.9: A constant function is Riemann integrable. Suppose $f(x) = c$ for all x on R. Then

$$cV(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq cV(R).$$

So f is integrable, and furthermore $\int_R f = cV(R)$.

The proofs of linearity and monotonicity are almost completely identical as the proofs from one variable. We therefore leave it as an exercise to prove the next two propositions.

Proposition 10.1.10 (Linearity)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathscr{R}(R)$ and $\alpha \in \mathbb{R}$.*

(i) αf is in $\mathscr{R}(R)$ and

$$\int_R \alpha f = \alpha \int_R f$$

(ii) $f + g$ is in $\mathscr{R}(R)$ and

$$\int_R (f+g) = \int_R f + \int_R g.$$

Proposition 10.1.11 (Monotonicity)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathscr{R}(R)$ and let $f(x) \leq g(x)$ for all $x \in R$. Then*

$$\int_R f \leq \int_R g.$$

Again for simplicity if $f\colon S \to \mathbb{R}$ is a function and $R \subset S$ is a closed rectangle, then if the restriction $f|_R$ is integrable we say f is integrable on R, or $f \in \mathscr{R}(R)$ and we write

$$\int_R f := \int_R f|_R.$$

Proposition 10.1.12. *For a closed rectangle $S \subset \mathbb{R}^n$, if $f\colon S \to \mathbb{R}$ is integrable and $R \subset S$ is a closed rectangle, then f is integrable over R.*

Proof. Given $\varepsilon > 0$, we find a partition P such that $U(P, f) - L(P, f) < \varepsilon$. By making a refinement of P we can assume that the endpoints of R are in P, or in other words, R is a union of subrectangles of P. Then the subrectangles of P divide into two collections, ones that are subsets of R and ones whose intersection with the interior of R is empty. Suppose that $R_1, R_2 \ldots, R_K$ be the subrectangles that are subsets of R and $R_{K+1},\ldots,R_N$ be the rest. Let $\widetilde{P}$ be the partition of R composed of those subrectangles of P contained in R. Then using the same notation as before.

$$\begin{aligned}\varepsilon > U(P,f)-L(P,f) &= \sum_{k=1}^{K}(M_k-m_k)V(R_k)+\sum_{k=K+1}^{N}(M_k-m_k)V(R_k)\\ &\geq \sum_{k=1}^{K}(M_k-m_k)V(R_k) = U(\widetilde{P},f|_R)-L(\widetilde{P},f|_R)\end{aligned}$$

Therefore $f|_R$ is integrable. □

10.1.4 Integrals of continuous functions

Later we will prove a much more general result, but it is useful to start with continuous functions only and prove that continuous functions are integrable. Before we get to continuous functions, let us state the following proposition, which has a very easy proof, but it is useful to emphasize as a technique.

Proposition 10.1.13. *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f\colon R \to \mathbb{R}$ a bounded function. If for every $\varepsilon > 0$, there exists a partition P of R such that*

$$U(P,f)-L(P,f) < \varepsilon,$$

then $f \in \mathscr{R}(R)$.

Proof. Given an $\varepsilon > 0$ find P as in the hypothesis. Then

$$\overline{\int_R} f - \underline{\int_R} f \leq U(P,f)-L(P,f) < \varepsilon.$$

As $\overline{\int_R} f \geq \underline{\int_R} f$ and the above holds for every $\varepsilon > 0$, we conclude $\overline{\int_R} f = \underline{\int_R} f$ and $f \in \mathscr{R}(R)$. □

We say a rectangle $R = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$ has *longest side at most α* if $b_k - a_k \leq \alpha$ for all $k = 1,2,\ldots,n$.

Proposition 10.1.14. *If a rectangle $R \subset \mathbb{R}^n$ has longest side at most α. Then for any $x,y \in R$,*

$$\|x-y\| \leq \sqrt{n}\,\alpha.$$

Proof.

$$\begin{aligned}\|x-y\| &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}\\ &\leq \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+\cdots+(b_n-a_n)^2}\\ &\leq \sqrt{\alpha^2+\alpha^2+\cdots+\alpha^2} = \sqrt{n}\,\alpha.\end{aligned}$$

□

Theorem 10.1.15. *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f \colon R \to \mathbb{R}$ a continuous function, then $f \in \mathscr{R}(R)$.*

Proof. The proof is analogous to the one variable proof with some complications. The set R is closed and bounded and hence compact. So f is not just continuous, but in fact uniformly continuous by Proposition **??**. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $\|x-y\| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{V(R)}$.

Let P be a partition of R such that longest side of any subrectangle is strictly less than $\frac{\delta}{\sqrt{n}}$. Then for all $x, y \in R_k$ for a subrectangle R_k of P we have, by the proposition above, $\|x-y\| < \sqrt{n}\frac{\delta}{\sqrt{n}} = \delta$. Therefore

$$f(x) - f(y) \leq |f(x) - f(y)| < \frac{\varepsilon}{V(R)}.$$

As f is continuous on R_k, it attains a maximum and a minimum on this interval. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_k$ and $f(y) = m_k$ in the notation from the definition of the integral. Therefore,

$$M_i - m_i = f(x) - f(y) < \frac{\varepsilon}{V(R)}.$$

And so

$$\begin{aligned} U(P,f) - L(P,f) &= \left(\sum_{k=1}^{N} M_k V(R_k)\right) - \left(\sum_{k=1}^{N} m_k V(R_k)\right) \\ &= \sum_{k=1}^{N} (M_k - m_k) V(R_k) \\ &< \frac{\varepsilon}{V(R)} \sum_{k=1}^{N} V(R_k) = \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary,

$$\overline{\int_a^b} f = \underline{\int_a^b} f,$$

and f is Riemann integrable on R. $\square$

10.1.5 Integration of functions with compact support

Let $U \subset \mathbb{R}^n$ be an open set and $f \colon U \to \mathbb{R}$ be a function. We say the *support* of f is the set

$$\operatorname{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}}.$$

That is, the support is the closure of the set of points where the function is nonzero. The closure is in U, that is in particular $\operatorname{supp}(f) \subset U$. So for a point $x \in U$ not in the support we have that f is constantly zero in a whole neighbourhood of x.

A function f is said to have *compact support* if $\operatorname{supp}(f)$ is a compact set. We will mostly consider the case when $U = \mathbb{R}^n$. In light of the following exercise, this is not an oversimplification.

Exercise* 10.1.1*: *Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}$ is continuous and of compact support. Show that the function $\widetilde{f} \colon \mathbb{R}^n \to \mathbb{R}$*

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous.

Proposition 10.1.16. *Suppose $f \colon \mathbb{R}^n \to \mathbb{R}$ be a function with compact support. If R is a closed rectangle such that $\operatorname{supp}(f) \subset R^o$ where R^o is the interior of R, and f is integrable over R, then for any other closed rectangle S with $\operatorname{supp}(f) \subset S^o$, the function f is integrable over S and*

$$\int_S f = \int_R f.$$

Proof. The intersection of closed rectangles is again a closed rectangle (or empty). Therefore we can take $\widetilde{R} = R \cap S$ be the intersection of all rectangles containing $\operatorname{supp}(f)$. If $\widetilde{R}$ is the empty set, then $\operatorname{supp}(f)$ is the empty set and f is identically zero and the proposition is trivial. So suppose that $\widetilde{R}$ is nonempty. As $\widetilde{R} \subset R$, we know that f is integrable over $\widetilde{R}$. Furthermore $\widetilde{R} \subset S$. Given $\varepsilon > 0$, take $\widetilde{P}$ to be a partition of $\widetilde{R}$ such that

$$U(\widetilde{P}, f|_{\widetilde{R}}) - L(\widetilde{P}, f|_{\widetilde{R}}) < \varepsilon.$$

Now add the endpoints of S to $\widetilde{P}$ to create a new partition P. Note that the subrectangles of $\widetilde{P}$ are subrectangles of P as well. Let $R_1, R_2, \ldots, R_K$ be the subrectangles of $\widetilde{P}$ and $R_{K+1}, \ldots, R_N$ the new subrectangles. Note that since $\operatorname{supp}(f) \subset \widetilde{R}$, then for $k = K+1, \ldots, N$ we have $\operatorname{supp}(f) \cap R_k = \emptyset$. In other words f is identically zero on R_k. Therefore in the notation used previously we have

$$\begin{aligned} U(P, f|_S) - L(P, f|_S) &= \sum_{k=1}^{K} (M_k - m_k) V(R_k) + \sum_{k=K+1}^{N} (M_k - m_k) V(R_k) \\ &= \sum_{k=1}^{K} (M_k - m_k) V(R_k) + \sum_{k=K+1}^{N} (0) V(R_k) \\ &= U(\widetilde{P}, f|_{\widetilde{R}}) - L(\widetilde{P}, f|_{\widetilde{R}}) < \varepsilon. \end{aligned}$$

Similarly we have that $L(P, f|_S) = L(\widetilde{P}, f_{\widetilde{R}})$ and therefore

$$\int_S f = \int_{\widetilde{R}} f.$$

Since $\widetilde{R} \subset R$ we also get $\int_R f = \int_{\widetilde{R}} f$, or in other words $\int_R f = \int_S f$. $\square$

Because of this proposition, when $f\colon \mathbb{R}^n \to \mathbb{R}$ has compact support and is integrable over a rectangle R containing the support we write

$$\int f := \int_R f \qquad \text{or} \qquad \int_{\mathbb{R}^n} f := \int_R f.$$

For example if f is continuous and of compact support then $\int_{\mathbb{R}^n} f$ exists.

10.1.6 Exercises

***Exercise* 10.1.2:** *Prove Proposition 10.1.10.*

***Exercise* 10.1.3:** *Suppose that R is a rectangle with the length of one of the sides equal to 0. Show that for any function f show that $f \in \mathscr{R}(R)$ and $\int_R f = 0$.*

***Exercise* 10.1.4:** *Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose that $f\colon R \to \mathbb{R}$ is in $\mathscr{R}(R)$. Show that $f \in \mathscr{R}(R')$.*

***Exercise* 10.1.5:** *Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose that $f\colon R \to \mathbb{R}$ is in $\mathscr{R}(R')$ and $f(x) = 0$ for $x \notin R'$. Show that $f \in \mathscr{R}(R)$ and*

$$\int_{R'} f = \int_R f.$$

Hint: see the previous exercise.

***Exercise* 10.1.6:** *Suppose that $R' \subset \mathbb{R}^n$ and $R'' \subset \mathbb{R}^n$ are two rectangles such that $R = R' \cup R''$ is a rectangle, and $R' \cap R''$ is rectangle with one of the sides having length 0 (that is $V(R' \cap R'') = 0$). Let $f\colon R \to \mathbb{R}$ be a function such that $f \in \mathscr{R}(R')$ and $f \in \mathscr{R}(R'')$. Show that $f \in \mathscr{R}(R)$ and*

$$\int_R f = \int_{R'} f + \int_{R''} f.$$

Hint: see previous exercise.

***Exercise* 10.1.7:** *Prove a stronger version of Proposition 10.1.16. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function with compact support. Prove that if R is a closed rectangle such that* $\operatorname{supp}(f) \subset R$ *and f is integrable over R, then for any other closed rectangle S with* $\operatorname{supp}(f) \subset S$*, the function f is integrable over S and $\int_S f = \int_R f$.* Hint: notice that now the new rectangles that you add as in the proof can intersect $\operatorname{supp}(f)$ on their boundary.

***Exercise* 10.1.8:** *Suppose that R and S are closed rectangles. Let $f(x) := 1$ if $x \in R$ and $f(x) = 0$ otherwise. Show that f is integrable over S and compute $\int_S f$.*

***Exercise* 10.1.9:** *Let $R = [0,1] \times [0,1] \subset \mathbb{R}^2$.*
a) Suppose $f \colon R \to \mathbb{R}$ is defined by

$$f(x,y) := \begin{cases} 1 & \textit{if } x = y, \\ 0 & \textit{else.} \end{cases}$$

Show that $f \in \mathscr{R}(R)$ and compute $\int_R f$.
b) Suppose $f \colon R \to \mathbb{R}$ is defined by

$$f(x,y) := \begin{cases} 1 & \textit{if } x \in \mathbb{Q} \textit{ or } y \in \mathbb{Q}, \\ 0 & \textit{else.} \end{cases}$$

Show that $f \notin \mathscr{R}(R)$.

10.2 Iterated integrals and Fubini theorem

Note: ??? lectures

The Riemann integral in several variables is hard to compute from the definition. For one-dimensional Riemann integral we have the fundamental theorem of calculus and we can compute many integrals without having to appeal to the definition of the integral. We will rewrite a a Riemann integral in several variables into several one-dimensional Riemann integrals by iterating. However, if $f \colon [0,1]^2 \to \mathbb{R}$ is a Riemann integrable function, it is not immediately clear if the three expressions

$$\int_{[0,1]^2} f, \qquad \int_0^1 \int_0^1 f(x,y)\,dx\,dy, \qquad \text{and} \qquad \int_0^1 \int_0^1 f(x,y)\,dy\,dx$$

are equal, or if the last two are even well-defined.

Example 10.2.1: Define

$$f(x,y) := \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is Riemann integrable on $R := [0,1]^2$ and $\int_R f = 0$. Furthermore, $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$. However

$$\int_0^1 f(1/2,y)\,dy$$

does not exist, so we cannot even write $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$.

Proof: Let us start with integrability of f. We simply take the partition of $[0,1]^2$ where the partition in the x direction is $\{0, 1/2-\varepsilon, 1/2+\varepsilon, 1\}$ and in the y direction $\{0,1\}$. The subrectangles of the partition are

$$R_1 := [0, 1/2-\varepsilon] \times [0,1], \qquad R_2 := [1/2-\varepsilon, 1/2+\varepsilon] \times [0,1], \qquad R_3 := [1/2+\varepsilon, 1] \times [0,1].$$

We have $m_1 = M_1 = 0$, $m_2 = 0$, $M_2 = 1$, and $m_3 = M_3 = 0$. Therefore,

$$L(P,f) = m_1(1/2-\varepsilon)\cdot 1 + m_2(2\varepsilon)\cdot 1 + m_3(1/2-\varepsilon)\cdot 1 = 0,$$

and

$$U(P,f) = M_1(1/2-\varepsilon)\cdot 1 + M_2(2\varepsilon)\cdot 1 + M_3(1/2-\varepsilon)\cdot 1 = 2\varepsilon.$$

The upper and lower sum are arbitrarily close and the lower sum is always zero, so the function is integrable and $\int_R f = 0$.

For any y, the function that takes x to $f(x,y)$ is zero except perhaps at a single point $x = 1/2$. We know that such a function is integrable and $\int_0^1 f(x,y)\,dx = 0$. Therefore, $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$.

However if $x = 1/2$, the function that takes y to $f(1/2,y)$ is the nonintegrable function that is 1 on the rationals and 0 on the irrationals. See Example **??**.

We will solve this problem of undefined inside integrals by using the upper and lower integrals, which are always defined.

We split $\mathbb{R}^{n+m}$ into two parts. That is, we write the coordinates on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ as (x,y) where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a function $f(x,y)$ we write

$$f_x(y) := f(x,y)$$

when x is fixed and we wish to speak of the function in terms of y. We write

$$f^y(x) := f(x,y)$$

when y is fixed and we wish to speak of the function in terms of x.

Theorem 10.2.2 (Fubini version A)**.** *Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f \colon R \times S \to \mathbb{R}$ be integrable. The functions $g \colon R \to \mathbb{R}$ and $h \colon R \to \mathbb{R}$ defined by*

$$g(x) := \underline{\int_S} f_x \qquad \text{and} \qquad h(x) := \overline{\int_S} f_x$$

are integrable over R and

$$\int_R g = \int_R h = \int_{R \times S} f.$$

In other words

$$\int_{R\times S} f = \int_R \left(\underline{\int_S} f(x,y)\,dy \right) dx = \int_R \left(\overline{\int_S} f(x,y)\,dy \right) dx.$$

If it turns out that f_x is integrable for all x, for example when f is continuous, then we obtain the more familiar

$$\int_{R\times S} f = \int_R \int_S f(x,y)\,dy\,dx.$$

Proof. Let P be a partition of R and P' be a partition of S. Let $R_1, R_2, \ldots, R_N$ be the subrectangles of P and $R'_1, R'_2, \ldots, R'_K$ be the subrectangles of P'. Then $P \times P'$ is the partition whose subrectangles are $R_j \times R'_k$ for all $1 \leq j \leq N$ and all $1 \leq k \leq K$.

Let

$$m_{j,k} := \inf_{(x,y) \in R_j \times R'_k} f(x,y).$$

We notice that $V(R_j \times R'_k) = V(R_j)V(R'_k)$ and hence

$$L(P \times P', f) = \sum_{j=1}^{N} \sum_{k=1}^{K} m_{j,k} V(R_j \times R'_k) = \sum_{j=1}^{N} \left(\sum_{k=1}^{K} m_{j,k} V(R'_k) \right) V(R_j).$$

If we let

$$m_k(x) := \inf_{y \in R'_k} f(x,y) = \inf_{y \in R'_k} f_x(y),$$

then of course if $x \in R_j$ then $m_{j,k} \leq m_k(x)$. Therefore

$$\sum_{k=1}^{K} m_{j,k} V(R'_k) \leq \sum_{k=1}^{K} m_k(x) V(R'_k) = L(P', f_x) \leq \underline{\int_S} f_x = g(x).$$

As we have the inequality for all $x \in R_j$ we have

$$\sum_{k=1}^{K} m_{j,k} V(R'_k) \leq \inf_{x \in R_j} g(x).$$

We thus obtain

$$L(P \times P', f) \leq \sum_{j=1}^{N} \left(\inf_{x \in R_j} g(x) \right) V(R_j) = L(P, g).$$

Similarly $U(P \times P', f) \geq U(P, h)$, and the proof of this inequality is left as an exercise.

Putting this together we have

$$L(P \times P', f) \leq L(P, g) \leq U(P, g) \leq U(P, h) \leq U(P \times P', f).$$

And since f is integrable, it must be that g is integrable as

$$U(P, g) - L(P, g) \leq U(P \times P', f) - L(P \times P', f),$$

and we can make the right hand side arbitrarily small. Furthermore as $L(P \times P', f) \leq L(P, g) \leq U(P \times P', f)$ we must have that $\int_R g = \int_{R \times S} f$.

Similarly we have

$$L(P \times P', f) \leq L(P, g) \leq L(P, h) \leq U(P, h) \leq U(P \times P', f),$$

and hence

$$U(P, h) - L(P, h) \leq U(P \times P', f) - L(P \times P', f).$$

So if f is integrable so is h, and as $L(P \times P', f) \leq L(P, h) \leq U(P \times P', f)$ we must have that $\int_R h = \int_{R \times S} f$. $\square$

We can also do the iterated integration in opposite order. The proof of this version is almost identical to version A, and we leave it as an exercise to the reader.

Theorem 10.2.3 (Fubini version B)**.** *Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f \colon R \times S \to \mathbb{R}$ be integrable. The functions $g \colon S \to \mathbb{R}$ and $h \colon S \to \mathbb{R}$ defined by*

$$g(y) := \underline{\int_R} f^y \qquad \text{and} \qquad h(y) := \overline{\int_R} f^y$$

are integrable over S and

$$\int_S g = \int_S h = \int_{R \times S} f.$$

That is we also have

$$\int_{R \times S} f = \int_S \left(\underline{\int_R} f(x,y)\, dx \right) dy = \int_S \left(\overline{\int_R} f(x,y)\, dx \right) dy.$$

Next suppose that f_x and f^y are integrable for simplicity. For example, suppose that f is continuous. Then by putting the two versions together we obtain the familiar

$$\int_{R \times S} f = \int_R \int_S f(x,y)\, dy\, dx = \int_S \int_R f(x,y)\, dx\, dy.$$

Often the Fubini theorem is stated in two dimensions for a continuous function $f \colon R \to \mathbb{R}$ on a rectangle $R = [a,b] \times [c,d]$. Then the Fubini theorem states that

$$\int_R f = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

And the Fubini theorem is commonly thought of as the theorem that allows us to swap the order of iterated integrals.

Repeatedly applying Fubini theorem gets us the following corollary: Let $R := [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n] \subset \mathbb{R}^n$ be a closed rectangle and let $f \colon R \to \mathbb{R}$ be continuous. Then

$$\int_R f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1,x_2,\ldots,x_n)\, dx_n\, dx_{n-1} \cdots dx_1.$$

Clearly we can also switch the order of integration to any order we please. We can also relax the continuity requirement by making sure that all the intermediate functions are integrable, or by using upper or lower integrals.

10.2.1 Exercises

Exercise* 10.2.1*: *Compute $\int_0^1 \int_{-1}^1 x e^{xy}\, dx\, dy$ in a simple way.*

Exercise* 10.2.2*: *Prove the assertion $U(P \times P', f) \geq U(P,h)$ from the proof of Theorem 10.2.2.*

Exercise* 10.2.3*: *Prove Theorem 10.2.3.*

***Exercise* 10.2.4:** *Let $R = [a,b] \times [c,d]$ and $f(x,y) := g(x)h(y)$ for two continuous functions $g\colon [a,b] \to \mathbb{R}$ and $h\colon [a,b] \to \mathbb{R}$. Prove*

$$\int_R f = \left(\int_a^b g\right)\left(\int_c^d h\right).$$

***Exercise* 10.2.5:** *Compute*

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy \qquad \textit{and} \qquad \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

You will need to interpret the integrals as improper, that is, the limit of $\int_\varepsilon^1$ as $\varepsilon \to 0$.

***Exercise* 10.2.6:** *Suppose $f(x,y) := g(x)$ where $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable. Show that f is Riemann integrable for any $R = [a,b] \times [c,d]$ and*

$$\int_R f = (d-c)\int_a^b g.$$

***Exercise* 10.2.7:** *Define $f\colon [-1,1] \times [0,1] \to \mathbb{R}$ by*

$$f(x,y) := \begin{cases} x & \textit{if } y \in \mathbb{Q}, \\ 0 & \textit{else.} \end{cases}$$

Show
a) $\int_0^1 \int_{-1}^1 f(x,y)\,dx\,dy$ exists, but $\int_{-1}^1 \int_0^1 f(x,y)\,dy\,dx$ does not.
b) Compute $\int_{-1}^1 \overline{\int_0^1} f(x,y)\,dy\,dx$ and $\int_{-1}^1 \underline{\int_0^1} f(x,y)\,dy\,dx$.
c) Show f is not Riemann integrable on $[-1,1] \times [0,1]$ (use Fubini).

10.3 Outer measure and null sets

Note: ??? lectures

10.3.1 Outer measure and null sets

Before we characterize all Riemann integrable functions, we need to make a slight detour. We introduce a way of measuring the size of sets in $\mathbb{R}^n$.

Definition 10.3.1. Let $S \subset \mathbb{R}^n$ be a subset. Define the *outer measure* of S as

$$m^*(S) := \inf \sum_{j=1}^{\infty} V(R_j),$$

where the infimum is taken over all sequences $\{R_j\}$ of open rectangles such that $S \subset \bigcup_{j=1}^{\infty} R_j$. In particular S is of *measure zero* or a *null set* if $m^*(S) = 0$.

We will only need measure zero sets and so we focus on these. Note that S is of measure zero if for every $\varepsilon > 0$ there exist a sequence of open rectangles $\{R_j\}$ such that

$$S \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \varepsilon. \tag{10.2}$$

Furthermore, if S is measure zero and $S' \subset S$, then S' is of measure zero. We can in fact use the same exact rectangles.

We can also use balls and it is sometimes more convenient. In fact we can choose balls no bigger than a fixed radius.

Proposition 10.3.2. *Let $\delta > 0$ be given. A set $S \subset \mathbb{R}^n$ is measure zero if and only if for every $\varepsilon > 0$, there exists a sequence of open balls $\{B_j\}$, where the radius of B_j is $r_j < \delta$ such that*

$$S \subset \bigcup_{j=1}^{\infty} B_j \qquad \textit{and} \qquad \sum_{j=1}^{\infty} r_j^n < \varepsilon.$$

Note that the "volume" of B_j is proportional to r_j^n.

Proof. First note that if R is a (closed or open) cube (rectangle with all sides equal) of side s, then R is contained in a closed ball of radius $\sqrt{n}\,s$ by Proposition 10.1.14, and therefore in an open ball of size $2\sqrt{n}\,s$.

Let s be a number that is less than the smallest side of R and also so that $2\sqrt{n}\,s < \delta$. We claim R is contained in a union of closed cubes $C_1, C_2, \ldots, C_k$ of sides s such that

$$\sum_{j=1}^{k} V(C_j) \leq 2^n V(R).$$

It is clearly true (without the 2^n) if R has sides that are integer multiples of s. So if a side is of length $(\ell+\alpha)s$, for $\ell \in \mathbb{N}$ and $0 \leq \alpha < 1$, then $(\ell+\alpha)s \leq 2\ell s$. Increasing the side to $2\ell s$ we obtain a new larger rectangle of volume at most 2^n times larger, but whose sides are multiples of s.

So suppose that there exist $\{R_j\}$ as in the definition such that (10.2) is true. As we have seen above, we can choose closed cubes $\{C_k\}$ with C_k of side s_k as above that cover all the rectangles $\{R_j\}$ and so

$$\sum_{k=1}^{\infty} s_k^n = \sum_{k=1}^{\infty} V(C_k) \leq 2^n \sum_{j=1}^{\infty} V(R_k) < 2^n \varepsilon.$$

Covering C_k with balls B_k of radius $r_k = 2\sqrt{n}\,s_k$ we obtain

$$\sum_{k=1}^{\infty} r_k^n < 2^{2n} n\varepsilon.$$

And as $S \subset \bigcup_j R_j \subset \bigcup_k C_k \subset \bigcup_k B_k$, we are finished.

Suppose that we have the ball condition above for some $\varepsilon > 0$. Without loss of generality assume that all $r_j < 1$. Each B_j is contained a in a cube (rectangle with all sides equal) R_j of side $2r_j$. So $V(R_j) = (2r_j)^n < 2^n r_j$. Therefore

$$S \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} 2^n r_j < 2^n \varepsilon. \qquad \square$$

In fact the definition of outer measure could have been done with open balls as well, not just null sets. Although we leave this generalization to the reader.

10.3.2 Examples and basic properties

Example 10.3.3: The set $\mathbb{Q}^n \subset \mathbb{R}^n$ of points with rational coordinates is a set of measure zero.

Proof: The set $\mathbb{Q}^n$ is countable and therefore let us write it as a sequence $q_1, q_2, \ldots$. For each q_j find an open rectangle R_j with $q_j \in R_j$ and $V(R_j) < \varepsilon 2^{-j}$. Then

$$\mathbb{Q}^n \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \varepsilon.$$

In fact, the example points to a more general result.

Proposition 10.3.4. *A countable union of measure zero sets is of measure zero.*

Proof. Suppose

$$S = \bigcup_{j=1}^{\infty} S_j$$

where S_j are all measure zero sets. Let $\varepsilon > 0$ be given. For each j there exists a sequence of open rectangles $\{R_{j,k}\}_{k=1}^{\infty}$ such that

$$S_j \subset \bigcup_{k=1}^{\infty} R_{j,k}$$

and

$$\sum_{k=1}^{\infty} V(R_{j,k}) < 2^{-j}\varepsilon.$$

Then

$$S \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} R_{j,k}.$$

As $V(R_{j,k})$ is always positive, the sum over all j and k can be done in any order. In particular, it can be done as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} V(R_{j,k}) < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon. \qquad \square$$

The next example is not just interesting, it will be useful later.

Example 10.3.5: Let $P := \{x \in \mathbb{R}^n : x_k = c\}$ for a fixed $k = 1, 2, \ldots, n$ and a fixed constant $c \in \mathbb{R}$. Then P is of measure zero.

Proof: First fix s and let us prove that

$$P_s := \{x \in \mathbb{R}^n : x_k = c, \left|x_j\right| \leq s \text{ for all } j \neq k\}$$

is of measure zero. Given any $\varepsilon > 0$ define the open rectangle

$$R := \{x \in \mathbb{R}^n : c - \varepsilon < x_k < c + \varepsilon, \left|x_j\right| < s + 1 \text{ for all } j \neq k\}$$

It is clear that $P_s \subset R$. Furthermore

$$V(R) = 2\varepsilon\bigl(2(s+1)\bigr)^{n-1}.$$

As s is fixed, we can make $V(R)$ arbitrarily small by picking ε small enough.

Next we note that

$$P = \bigcup_{j=1}^{\infty} P_j$$

and a countable union of measure zero sets is measure zero.

Example 10.3.6: If $a < b$, then $m^*([a,b]) = b - a$.

Proof: In the case of $\mathbb{R}$, open rectangles are open intervals. Since $[a,b] \subset (a-\varepsilon, b+\varepsilon)$ for all $\varepsilon > 0$. Hence, $m^*([a,b]) \leq b - a$.

Let us prove the other inequality. Suppose that $\{(a_j,b_j)\}$ are open intervals such that

$$[a,b] \subset \bigcup_{j=1}^{\infty} (a_j,b_j).$$

We wish to bound $\sum(b_j - a_j)$ from below. Since $[a,b]$ is compact, then there are only finitely many open intervals that still cover $[a,b]$. As throwing out some of the intervals only makes the sum smaller, we only need to take the finite number of intervals still covering $[a,b]$. If $(a_i,b_i) \subset (a_j,b_j)$, then we can throw out (a_i,b_i) as well. Therefore we have $[a,b] \subset \bigcup_{j=1}^{k}(a_j,b_j)$ for some k, and we assume that the intervals are sorted such that $a_1 < a_2 < \cdots < a_k$. Note that since (a_2,b_2) is not contained in (a_1,b_1) we have that $a_1 < a_2 < b_1 < b_2$. Similarly $a_j < a_{j+1} < b_j < b_{j+1}$. Furthermore, $a_1 < a$ and $b_k > b$. Thus,

$$m^*([a,b]) \geq \sum_{j=1}^{k} (b_j - a_j) \geq \sum_{j=1}^{k-1} (a_{j+1} - a_j) + (b_k - a_k) = b_k - a_1 > b - a.$$

Proposition 10.3.7. *Suppose $E \subset \mathbb{R}^n$ is a compact set of measure zero. Then for every $\varepsilon > 0$, there exist finitely many open rectangles $R_1, R_2, \ldots, R_k$ such that*

$$E \subset R_1 \cup R_2 \cup \cdots \cup R_k \qquad \textit{and} \qquad \sum_{j=1}^{k} V(R_j) < \varepsilon.$$

Also for any $\delta > 0$, there exist finitely many open balls $B_1, B_2, \ldots, B_k$ of radii $r_1, r_2, \ldots, r_k < \delta$ such that

$$E \subset B_1 \cup B_2 \cup \cdots \cup B_k \qquad \textit{and} \qquad \sum_{j=1}^{k} r_j^n < \varepsilon.$$

Proof. Find a sequence of open rectangles $\{R_j\}$ such that

$$E \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \varepsilon.$$

By compactness, finitely many of these rectangles still contain E. That is, there is some k such that $E \subset R_1 \cup R_2 \cup \cdots \cup R_k$. Hence

$$\sum_{j=1}^{k} V(R_j) \leq \sum_{j=1}^{\infty} V(R_j) < \varepsilon.$$

The proof that we can choose balls instead of rectangles is left as an exercise. □

10.3.3 Images of null sets

Before we look at images of measure zero sets, let us see what a continuously differentiable function does to a ball.

Lemma 10.3.8. *Suppose $U \subset \mathbb{R}^n$ is an open set, $B \subset U$ is an open or closed ball of radius at most r, $f : B \to \mathbb{R}^n$ is continuously differentiable and suppose $\|f'(x)\| \leq M$ for all $x \in B$. Then $f(B) \subset B'$, where B' is a ball of radius at most Mr.*

Proof. Without loss of generality assume B is a closed ball. The ball B is convex, and hence via Proposition 8.4.2, that $\|f(x) - f(y)\| \leq M\|x-y\|$ for all x,y in B. In particular, suppose $B = C(y,r)$, then $f(B) \subset C\bigl(f(y), Mr\bigr)$. □

The image of a measure zero set using a continuous map is not necessarily a measure zero set. However if we assume the mapping is continuously differentiable, then the mapping cannot "stretch" the set too much. The proposition does not require compactness, and this is left as an exercise.

Proposition 10.3.9. *Suppose $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^n$ is a continuously differentiable mapping. If $E \subset U$ is a compact measure zero set, then $f(E)$ is measure zero.*

Proof. We must first handle a couple of techicalities. First let us replace U by a smaller open set to make $\|f'(x)\|$ bounded. At each point $x \in E$ pick an open ball $B(x, r_x)$ such that the closed ball $C(x, r_x) \subset U$. By compactness we only need to take finitely many points $x_1, x_2, \ldots, x_q$ to still conver E. Define

$$U' := \bigcup_{j=1}^{q} B(x_j, r_{x_j}), \qquad K := \bigcup_{j=1}^{q} C(x_j, r_{x_j}). \tag{10.3}$$

We have $E \subset U' \subset K \subset U$. The set K is compact. The function that takes x to $\|f'(x)\|$ is continuous, and therefore there exists an $M > 0$ such that $\|f'(x)\| \leq M$ for all $x \in K$.

So without loss of generality we may replace U by U' and from now on suppose that $\|f'(x)\| \leq M$ for all $x \in U$.

At each point $x \in E$ pick a ball $B(x, \delta_x)$ of maximum radius so that $B(x, \delta_x) \subset U$. Let $\delta = \inf_{x \in E} \delta_x$. Take a sequence $\{x_j\} \subset E$ so that $\delta_{x_j} \to \delta$. As E is compact, we can pick the sequence to be convergent to some $y \in E$. Once $\|x_j - y\| < \frac{\delta_y}{2}$, then $\delta_{x_j} > \frac{\delta_y}{2}$ by the triangle inequality. Therefore $\delta > 0$.

Given $\varepsilon > 0$, there exist balls $B_1, B_2, \ldots, B_k$ of radii $r_1, r_2, \ldots, r_k < \delta$ such that

$$E \subset B_1 \cup B_2 \cup \cdots \cup B_k \qquad \text{and} \qquad \sum_{j=1}^{k} r_j^n < \varepsilon.$$

Suppose $B'_1, B'_2, \ldots, B'_k$ are the balls of radius $Mr_1, Mr_2, \ldots, Mr_k$ from Lemma 10.3.8.

$$f(E) \subset f(B_1) \cup f(B_2) \cup \cdots \cup f(B_k) \subset B'_1 \cup B'_2 \cup \cdots \cup B'_k \qquad \text{and} \qquad \sum_{j=1}^{k} Mr_j^n < M\varepsilon. \qquad \square$$

10.3.4 Exercises

Exercise* 10.3.1*: *Finish the proof of Proposition 10.3.7, that is, show that you can use balls instead of rectangles.*

Exercise* 10.3.2*: *If $A \subset B$ then $m^*(A) \leq m^*(B)$.*

Exercise* 10.3.3*: *Show that if $R \subset \mathbb{R}^n$ is a closed rectangle then $m^*(R) = V(R)$.*

Exercise* 10.3.4*: *Prove a version of Proposition 10.3.9 without using compactness:*
a) Mimic the proof to first prove that the proposition holds only if E is relatively compact*; a set $E \subset U$ is relatively compact if the closure of E in the subspace topology on U is compact, or in other words if there exists a compact set K with $K \subset U$ and $E \subset K$.*
Hint: The bound on the size of the derivative still holds, but you may need to use countably many balls. Be careful as the closure of E need no longer be measure zero.
b) Now prove it for any null set E.
Hint: First show that $\{x \in U : d(x,y) \geq 1/m \text{ for all } y \notin U \text{ and } d(0,x) \leq m\}$ is a compact set for any $m > 0$.

Exercise* 10.3.5*: *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a continuously differentiable function. Let $G := \{(x,y) \in U \times \mathbb{R} : y = f(x)\}$ be the graph of f. Show that f is of measure zero.*

Exercise* 10.3.6*: *Given a closed rectangle $R \subset \mathbb{R}^n$, show that for any $\varepsilon > 0$ there exists a number $s > 0$ and finitely many open cubes $C_1, C_2, \ldots, C_k$ of side s such that $R \subset C_1 \cup C_2 \cup \cdots \cup C_k$ and*

$$\sum_{j=1}^{k} V(C_j) \leq V(R) + \varepsilon.$$

Exercise* 10.3.7*: *Show that there exists a number $k = k(n,r,\delta)$ depending only on n, r and δ such the following holds. Given $B(x,r) \subset \mathbb{R}^n$ and $\delta > 0$, there exist k open balls $B_1, B_2, \ldots, B_k$ of radius at most δ such that $B(x,r) \subset B_1 \cup B_2 \cup \cdots \cup B_k$. Note that you can find k that really only depends on n and the ratio δ/r.*

10.4 The set of Riemann integrable functions

Note: ??? lectures

10.4.1 Oscillation and continuity

Let $S \subset \mathbb{R}^n$ be a set and $f : S \to \mathbb{R}$ a function. Instead of just saying that f is or is not continuous at a point $x \in S$, we we need to be able to quantify how discontinuous f is at a function is at x. For any $\delta > 0$ define the oscillation of f on the δ-ball in subset topology that is $B_S(x,\delta) = B_{\mathbb{R}^n}(x,\delta) \cap S$ as

$$o(f,x,\delta) := \sup_{y \in B_S(x,\delta)} f(y) - \inf_{y \in B_S(x,\delta)} f(y) = \sup_{y_1,y_2 \in B_S(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr).$$

That is, $o(f,x,\delta)$ is the length of the smallest interval that contains the image $f\bigl(B_S(x,\delta)\bigr)$. Clearly $o(f,x,\delta) \geq 0$ and notice $o(f,x,\delta) \leq o(f,x,\delta')$ whenever $\delta < \delta'$. Therefore, the limit as $\delta \to 0$ from the right exists and we define the *oscillation* of a function f at x as

$$o(f,x) := \lim_{\delta \to 0^+} o(f,x,\delta) = \inf_{\delta > 0} o(f,x,\delta).$$

Proposition 10.4.1. *$f : S \to \mathbb{R}$ is continuous at $x \in S$ if and only if $o(f,x) = 0$.*

Proof. First suppose that f is continuous at $x \in S$. Then given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for $y \in B_S(x,\delta)$ we have $|f(x) - f(y)| < \varepsilon$. Therefore if $y_1, y_2 \in B_S(x,\delta)$ then

$$f(y_1) - f(y_2) = f(y_1) - f(x) - \bigl(f(y_2) - f(x)\bigr) < \varepsilon + \varepsilon = 2\varepsilon.$$

We take the supremum over y_1 and y_2

$$o(f,x,\delta) = \sup_{y_1,y_2 \in B_S(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr) \leq 2\varepsilon.$$

Hence, $o(x,f) = 0$.

On the other hand suppose that $o(x,f) = 0$. Given any $\varepsilon > 0$, find a $\delta > 0$ such that $o(f,x,\delta) < \varepsilon$. If $y \in B_S(x,\delta)$ then

$$|f(x) - f(y)| \leq \sup_{y_1,y_2 \in B_S(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr) = o(f,x,\delta) < \varepsilon. \qquad \square$$

Proposition 10.4.2. *Let $S \subset \mathbb{R}^n$ be closed, $f : S \to \mathbb{R}$, and $\varepsilon > 0$. The set $\{x \in S : o(f,x) \geq \varepsilon\}$ is closed.*

Proof. Equivalently we want to show that $G = \{x \in S : o(f,x) < \varepsilon\}$ is open in the subset topology. As $\inf_{\delta>0} o(f,x,\delta) < \varepsilon$, find a $\delta > 0$ such that

$$o(f,x,\delta) < \varepsilon$$

Take any $\xi \in B_S(x, \delta/2)$. Notice that $B_S(\xi, \delta/2) \subset B_S(x, \delta)$. Therefore,

$$o(f, \xi, \delta/2) = \sup_{y_1, y_2 \in B_S(\xi, \delta/2)} \bigl(f(y_1) - f(y_2)\bigr) \leq \sup_{y_1, y_2 \in B_S(x, \delta)} \bigl(f(y_1) - f(y_2)\bigr) = o(f, x, \delta) < \varepsilon.$$

So $o(f, \xi) < \varepsilon$ as well. As this is true for all $\xi \in B_S(x, \delta/2)$ we get that G is open in the subset topology and $S \setminus G$ is closed as is claimed. $\square$

10.4.2 The set of Riemann integrable functions

We have seen that continuous functions are Riemann integrable, but we also know that certain kinds of discontinuities are allowed. It turns out that as long as the discontinuities happen on a set of measure zero, the function is integrable and vice versa.

Theorem 10.4.3 (Riemann-Lebesgue)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f : R \to \mathbb{R}$ a bounded function. Then f is Riemann integrable if and only if the set of discontinuities of f is of measure zero (a null set).*

Proof. Let $S \subset R$ be the set of discontinuities of f. That is $S = \{x \in R : o(f, x) > 0\}$. The trick to this proof is to isolate the bad set into a small set of subrectangles of a partition. There are only finitely many subrectangles of a partition, so we will wish to use compactness. If S is closed, then it would be compact and we could cover it by small rectangles as it is of measure zero. Unfortunately, in general S is not closed so we need to work a little harder.

For every $\varepsilon > 0$, define

$$S_\varepsilon := \{x \in R : o(f, x) \geq \varepsilon\}.$$

By Proposition 10.4.2 S_ε is closed and as it is a subset of R which is bounded, S_ε is compact. Furthermore, $S_\varepsilon \subset S$ and S is of measure zero. Via Proposition 10.3.7 there are finitely many open rectangles $S_1, S_2, \ldots, S_k$ that cover S_ε and $\sum V(S_j) < \varepsilon$.

The set $T = R \setminus (S_1 \cup \cdots \cup S_k)$ is closed, bounded, and therefore compact. Furthermore for $x \in T$, we have $o(f, x) < \varepsilon$. Hence for each $x \in T$, there exists a small closed rectangle T_x with x in the interior of T_x, such that

$$\sup_{y \in T_x} f(y) - \inf_{y \in T_x} f(y) < 2\varepsilon.$$

The interiors of the rectangles T_x cover T. As T is compact there exist finitely many such rectangles $T_1, T_2, \ldots, T_m$ that covers T.

Now take all the rectangles $T_1, T_2, \ldots, T_m$ and $S_1, S_2, \ldots, S_k$ and construct a partition out of their endpoints. That is construct a partition P with subrectangles $R_1, R_2, \ldots, R_p$ such that every R_j is contained in T_ℓ for some ℓ or the closure of S_ℓ for some ℓ. Suppose we order the rectangles so that $R_1, R_2, \ldots, R_q$ are those that are contained in some T_ℓ, and $R_{q+1}, R_{q+2}, \ldots, R_p$ are the rest. In particular, we have

$$\sum_{j=1}^{q} V(R_j) \leq V(R) \qquad \text{and} \qquad \sum_{j=q+1}^{p} V(R_j) \leq \varepsilon.$$

Let m_j and M_j be the inf and sup over R_j as before. If $R_j \subset T_\ell$ for some ℓ, then $(M_j - m_j) < 2\varepsilon$. Let $B \in \mathbb{R}$ be such that $|f(x)| \leq B$ for all $x \in R$, so $(M_j - m_j) < 2B$ over all rectangles. Then

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{j=1}^{p} (M_j - m_j) V(R_j) \\ &= \left(\sum_{j=1}^{q} (M_j - m_j) V(R_j) \right) + \left(\sum_{j=q+1}^{p} (M_j - m_j) V(R_j) \right) \\ &\leq \left(\sum_{j=1}^{q} 2\varepsilon V(R_j) \right) + \left(\sum_{j=q+1}^{p} 2B V(R_j) \right) \\ &\leq 2\varepsilon V(R) + 2B\varepsilon = \varepsilon(2V(R) + 2B). \end{aligned}$$

Clearly, we can make the right hand side as small as we want and hence f is integrable.

For the other direction, suppose that f is Riemann integrable over R. Let S be the set of discontinuities again and now let

$$S_k := \{x \in R : o(f,x) \geq 1/k\}.$$

Fix a $k \in \mathbb{N}$. Given an $\varepsilon > 0$, find a partition P with subrectangles $R_1, R_2, \ldots, R_p$ such that

$$U(P,f) - L(P,f) = \sum_{j=1}^{p} (M_j - m_j) V(R_j) < \varepsilon$$

Suppose that $R_1, R_2, \ldots, R_p$ are order so that the interiors of $R_1, R_2, \ldots, R_q$ intersect S_k, while the interiors of $R_{q+1}, R_{q+2}, \ldots, R_p$ are disjoint from S_k. If $x \in R_j \cap S_k$ and x is in the interior of R_j so sufficiently small balls are completely inside R_j, then by definition of S_k we have $M_j - m_j \geq 1/k$. Then

$$\varepsilon > \sum_{j=1}^{p} (M_j - m_j) V(R_j) \geq \sum_{j=1}^{q} (M_j - m_j) V(R_j) \geq \frac{1}{k} \sum_{j=1}^{q} V(R_j)$$

In other words $\sum_{j=1}^{q} V(R_j) < k\varepsilon$. Let G be the set of all boundaries of all the subrectangles of P. The set G is of measure zero (see Example 10.3.5). Let R_j° denote the interior of R_j, then

$$S_k \subset R_1^\circ \cup R_2^\circ \cup \cdots \cup R_q^\circ \cup G.$$

As G can be covered by open rectangles arbitrarily small volume, S_k must be of measure zero. As

$$S = \bigcup_{k=1}^{\infty} S_k$$

and a countable union of measure zero sets is of measure zero, S is of measure zero. □

10.4.3 Exercises

Exercise* 10.4.1*: *Suppose that $f\colon (a,b)\times(c,d)\to\mathbb{R}$ is a bounded continuous function. Show that the integral of f over $R=[a,b]\times[c,d]$ makes sense and is uniquely defined. That is, set f to be anything on the boundary of R and compute the integral.*

Exercise* 10.4.2*: *Suppose $R\subset\mathbb{R}^n$ is a closed rectangle. Show that $\mathscr{R}(R)$, the set of Riemann integrable functions, is an algebra. That is, show that if $f,g\in\mathscr{R}(R)$ and $a\in\mathbb{R}$, then $af\in\mathscr{R}(R)$, $f+g\in\mathscr{R}(R)$ and $fg\in\mathscr{R}(R)$.*

Exercise* 10.4.3*: *Suppose $R\subset\mathbb{R}^n$ is a closed rectangle and $f\colon R\to\mathbb{R}$ is a bounded function which is zero except on a closed set $E\subset R$ of measure zero. Show that $\int_R f$ exists and compute it.*

Exercise* 10.4.4*: *Suppose $R\subset\mathbb{R}^n$ is a closed rectangle and $f\colon R\to\mathbb{R}$ and $g\colon R\to\mathbb{R}$ are two Riemann integrable functions. Suppose $f=g$ except for a closed set $E\subset R$ of measure zero. Show that $\int_R f=\int_R g$.*