Intertwining Ladder Representations for SU(p, q) into Dolbeault Cohomology

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In Honor of Jacques Carmona

ABSTRACT The positive spin ladder representations for $G = \mathrm{SU}(p, q)$ may be realized in a Fock space, in Dolbeault cohomology over $G/\mathrm{S}(\mathrm{U}(p, q-1) \times \mathrm{U}(1))$, and as certain holomorphic sections of a vector bundle over $G/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. A Penrose transform, also referred to as a double fibration transform, intertwines the Dolbeault model into the vector bundle model. By passing through the Fock space realization of the ladder representations, we invert the Penrose transform, and thus intertwine the ladder representations into Dolbeault cohomology.

Introduction

An important discovery in modern physics was the existence of various symmetry groups such as the Lorentz group and the conformal group which preserve (up to unitary or conformal equivalence) the Minkowski norm on four-dimensional space-time. These groups commute with the differential equations of mathematical physics, and so spaces of solutions to these equations correspond to invariant subspaces of certain vector spaces on which the group in question acts. This leads to an interest in constructing explicit models of unitary representations of various non-compact Lie groups and intertwining operators between various versions of these models. Desirable models might be ones on which the group acts in a natural, geometric way (i.e., by substitutions rather than an integral), on which the inner product may be computed explicitly, or in which the solutions to the corresponding differential equation of mathematical physics are exhibited explicitly.

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Here we consider the ladder representations of $\mathrm{SU}(p,q)$. The group $\mathrm{SU}(2,2)$ is locally isomorphic to the conformal group, mentioned above, and has a one-parameter family of irreducible representations, indexed by a half-integer called the *spin*, which occur on spaces of solutions to the massless field equations. These equations include the wave equation with spin 0 and Maxwell's equations with spin ± 1. The term "ladder representations" for this family may refer to the fact that the K-types of each such representation lie on a half-line, each occuring with multiplicity one. We index these representations by an integer equal to twice the spin, and we consider this family of representations for all of the groups $\mathrm{SU}(p,q)$.

Unitarity of the ladder representations of $\mathrm{SU}(2,2)$ was first proved by [JV]. Later constructions by Sternberg and Wolf [SW] and Blattner and Rawnsley [BR], expanding on earlier work of Carmona [C], identified these representations. The Hilbert space of indefinite harmonic forms appearing in the L^2-cohomology constructions of [C] and [BR] is essentially a Fock space (see [F] and [B]). The [BR] realization was used in [M] to give a geometric construction of the ladder representations of $G = \mathrm{SU}(p,q)$ on sections of vector bundles over G/K, where $K = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ is the maximal compact subgroup of G. An alternative construction by Davidson [D] realizes all of the highest weight modules for $\mathrm{SU}(p,q)$ as vector-valued functions over G/K. At about the same time, the inner product on indefinite harmonic forms first constructed by Carmona in [C] was generalized by Rawnsley, Schmid, and Wolf [RSW] to give a geometric construction of a much broader class of representations in Dolbeault cohomology over a homogeneous space G/H. Independently, Patton and Rossi [PR] constructed the ladder representations of $\mathrm{SU}(p,q)$ in Dolbeault cohomology over a homogeneous space G/H, where $H = \mathrm{S}(\mathrm{U}(p,q-1) \times \mathrm{U}(1))$, giving an alternative and more elementary proof of a special case of the result in [RSW]. More recently, $\mathcal{D}$-module methods have been used to construct a broad class of representations of a real, reductive Lie group as solutions of a family of invariant, linear differential equations on a manifold on which the group acts (see [KS]).

An important method for constructing intertwining operators between various models of Lie group representations is a double fibration transform, also called a Penrose transform. One of the first instances of this transform was used to construct solutions of Maxwell's equations in [EPW] (see also [BE], [E], [N], and [Z]). For $G = \mathrm{SU}(p,q)$, the transform Φ constructed in [M] from the Fock model to sections over G/K, though not based on a double fibration, is a direct-image mapping inspired by the Penrose transform. Patton and Rossi ([P], [PR])

$$\begin{tikzcd} & \text{Fock space on } \mathbf{C}^{p+q} \arrow[dl, "\Psi"'] \arrow[dr, shift left, "\Phi"] & \\ \text{cohomology on } G/H \arrow[rr, "P"] & & \text{sections over } G/K \arrow[ul, shift left, "\Phi^{-1}"] \end{tikzcd}$$

construct transforms Ψ, mapping the Fock space into Dolbeault cohomology over G/H, and P, a Penrose transform mapping cohomology over G/H into sections over G/K. Further, they show that P is one-to-one and that the diagram above commutes.

Here the inverse transform Φ^{-1} was constructed in [L1] and [L2] and used to explicitly describe inner product structures on the disk model of the ladder representations. This construction, however, is less than optimal, containing an *ad hoc* differential operator that is not G-equivariant. An alternative study of inner product structures on G/K for all but finitely many of the ladder representations was done in [DS]. Our goal in this work is to construct a transform P^{-1} in two steps. In the first step, we use [M] and analytical results from [L1] and [L2] to pull the disk realization back to the Fock realization. This step is completed for all but finitely many ladder representations, using a more natural and G-equivariant version of Φ^{-1} inspired by ideas of [DS]. Then, in the second step, we use the results of [P] and [PR] to map the Fock realization into Dolbeault cohomology. Once the transform is constructed, we verify that it inverts the Penrose transform on the disk realization, and that it annihilates K-finite vectors which are "outside" the image of the Penrose transform. It is important to emphasize that we invert P on the disk realization of the ladder representations, which is contained within, but is not equal to, the entire image of P.

The structure of the paper is as follows: in section 1, we describe the Fock space model of the ladder representations. In section 2 we introduce the sections over G/K, with G/K realized in the bounded model as a generalized unit disk, and describe the Fock-disk transform. In section 3 we describe the cohomology over G/H and introduce the Penrose transform. Our main results are contained in section 4, where we invert the Penrose transform.

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1 The Fock model for ladder representations

In this section we give a brief overview of the Fock realization for the oscillator representation of $G = \mathrm{SU}(p, q)$, and of how the oscillator representation decomposes into G-isotypic components. These isotypic components, which are of multiplicity one, are the *ladder representations* of $\mathrm{SU}(p, q)$. For more details, one may refer to [B],[BR],[D], or [M].

1.1 Notation

The following notation will be used in the sequel.

Definition 1.1. (a) For $r \in \mathbf{N}$, let $\mathbf{N}_0$ denote the set of nonnegative integers and let $\mathbf{N}_0^r$ denote the set of r-tuples of nonnegative integers.
(b) If $m = (m_1, m_2, \ldots, m_r) \in \mathbf{N}_0^r$, then $m! := m_1! m_2! \cdots m_r!$ and $|m| := m_1 + m_2 + \cdots + m_r$. If $z \in \mathbf{C}^r$, then $z^m := z_1^{m_1} z_2^{m_2} \cdots z_r^{m_r}$.
(c) We have a partial order $\leq$ on $\mathbf{N}_0^r$ given by $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, r\}$.
(d) For $r \in \mathbf{N}$ and $m \in \mathbf{N}_0$, let $\mathbf{N}_0^r(m)$ denote the elements of $\mathbf{N}_0^r$ of length m.
(e) For $p, q \in \mathbf{N}$, $z \in \mathbf{C}^{p+q}$, let $z_R := (z_1, \ldots, z_p)$ and $z_S := (z_{p+1}, \ldots, z_{p+q})$.

1.2 The Fock Space

Definition 1.2. *For positive integers p and q, put*

$$\mathcal{F} = \{f : \mathbf{C}^{p+q} \to \mathbf{C} \mid f \text{ holomorphic in } (z_R, \bar{z}_S) \text{ and } \int_{\mathbf{C}^{p+q}} |f(z)|^2 e^{-|z|^2}\, dm(z) < \infty\},$$

where m is Lebesgue measure normalized so that

$$\int_{\mathbf{C}^{p+q}} e^{-|z|^2}\, dm(z) = 1.$$

We assign an inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ to $\mathcal{F}$ defined by

$$\langle f, g \rangle_{\mathcal{F}} = \int_{\mathbf{C}^{p+q}} f(z) \overline{g(z)} e^{-|z|^2}\, dm(z). \tag{1.1}$$

The Hilbert space $\mathcal{F}$ is often referred to as a *Fock space*, on which V. Fock realized solutions to certain quantum mechanical systems (see [F]). One observes that the collection of polynomials in $z_R, \bar{z}_S$ is dense in $\mathcal{F}$. Also, given monomials $f(z) = z_R^\alpha \bar{z}_S^\beta$ and $g(z) = z_R^\gamma \bar{z}_S^\nu$ in $\mathcal{F}$, repeated integration by parts gives

$$\int_{\mathbf{C}^{p+q}} f(z) \overline{g(z)} e^{-|z|^2}\, dm(z) = \delta_{\alpha\gamma} \delta_{\beta\nu} \alpha! \beta!. \tag{1.2}$$

Thus we have the following lemma.

Lemma 1.3. *The collection of monomials $\{z_R^\alpha \bar{z}_S^\beta \mid \alpha \in \mathbf{N}_0^p, \beta \in \mathbf{N}_0^q\}$ forms an orthogonal basis for $\mathcal{F}$.*

Lemma 1.3 admits two useful corollaries, both of which may be proved by first using Lemma 1.3 to expand everything in sight into series, and then applying (1.1).

Corollary 1.4. *Suppose $f \in \mathcal{F}$ has series expansion*

$$f(z) = \sum_{\alpha \in \mathbf{N}_0^p, \beta \in \mathbf{N}_0^q} a_{\alpha,\beta} z_R^\alpha \bar{z}_S^\beta.$$

Then

$$\langle f, f\rangle_{\mathcal{F}} = \sum_{\alpha\in\mathbf{N}_0^p, \beta\in\mathbf{N}_0^q} |a_{\alpha,\beta}|^2 \alpha!\beta!.$$

Corollary 1.5. *The function* $B : \mathbf{C}^{p+q} \times \mathbf{C}^{p+q} \to \mathbf{C}$ *defined by*

$$B(z, w) = e^{z_R^* w_R + w_S^* z_S}$$

is a reproducing kernel for $\mathcal{F}$.

1.3 The Oscillator Representation

We define a unitary representation of $G = \mathrm{SU}(p, q)$ on the Fock space $\mathcal{F}$, called the *oscillator representation*, and show how it decomposes into isotypic components. For details regarding the construction of the oscillator representation, see [BR] or [D].

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the nondegenerate hermitian form on $\mathbf{C}^{p+q}$ of signature (p, q) determined by the matrix

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

and recall that G is the subgroup of $\mathrm{SL}(p + q, \mathbf{C})$ which preserves $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Theorem 1.6. [BR],[D] *Given* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ *and* $f \in \mathcal{F}$, *the formula*

$$[\sigma(g)f](z) = \det(\bar{D}) \int_{\mathbf{C}^{p+q}} f(g^{-1}w) e^{w_R^* z_R + z_S^* w_S} e^{\frac{1}{2}|g^{-1}w|^2} e^{\frac{1}{2}|w|^2}\, dm(w)$$

defines a continuous unitary representation of G *on* $\mathcal{F}$, *called the oscillator representation.*

The oscillator representation may be decomposed into isotypic components using a dual pairs argument. Observe that $\mathrm{U}(1)$ acts on $\mathcal{F}$ by right translation, and that this action commutes with the action of G. The isotypic components for G are given by decomposing $\mathcal{F}$ under the $\mathrm{U}(1)$-action.

Theorem 1.7. *The Fock space* $\mathcal{F}$ *decomposes into isotypic components of multiplicity one as*

$$\mathcal{F} = \bigoplus_{k\in\mathbf{Z}} \mathcal{F}_k,$$

where $\mathcal{F}_k = \{f \in \mathcal{F} \mid f(e^{-i\theta} z) = e^{ik\theta} f(z)\}$.

The representations $\mathcal{F}_k$ are called the *ladder representations* for G, since the highest weights for their K-types lie on a line in the weight lattice. Observe that the elements of $\mathcal{F}_k$ are S^1-homogeneous of degree k. In the sequel, when we speak of homogeneity, we mean S^1-homogeneity.

2 Realizing ladder representations over G/K

In this section we realize the ladder representations of G over G/K, where

$$K = \left\{ g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid g \in G \right\} \cong \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$$

is a maximal compact subgroup of G. This is accomplished via an intertwining integral transform constructed in [M].

2.1 The Generalized Unit Disk and G/K

We introduce a bounded model for G/K, called the *generalized unit disk*. We also give a natural action of G on polynomial-valued functions on G/K.

Definition 2.1. *For positive integers p and q, put*

$$\mathcal{D}_{p,q} = \{\zeta \in \mathbf{C}^{p \times q} \mid I_q - \zeta^* \zeta \gg 0\}.$$

The set $\mathcal{D}_{p,q}$, called the *generalized unit disk*, is a bounded domain in $\mathbf{C}^{p\times q}$ and is a Siegel domain of genus II. In case $p = 1$ or $q = 1$, $\mathcal{D}_{p,q}$ is simply a unit ball. Observe that $\mathcal{D}_{p,q}$ parametrizes the set M_q of negative q-planes in $\mathbf{C}^{p+q}$ via

$$\zeta \in \mathcal{D}_{p,q} \mapsto \text{col span} \begin{pmatrix} \zeta \\ I_q \end{pmatrix} \in M_q.$$

In turn, by Witt's theorem (see [La]) G acts transitively on M_q while the negative plane corresponding to $0 \in \mathcal{D}_{p,q}$ is stabilized by K. Thus $M_q \cong G/K$, and so $\mathcal{D}_{p,q}$ parametrizes G/K. In the sequel, we use $\mathcal{D}_{p,q}$ as our preferred model for G/K.

Now we work toward defining an action of G on polynomial-valued functions on G/K, culminating in Definition 2.4. First, we note that the canonical action of G on G/K induces an action of G on $\mathcal{D}_{p,q}$ by fractional linear transformations. Namely, if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $\zeta \in \mathcal{D}_{p,q}$, then

$$g.\zeta = (A\zeta + B)(C\zeta + D)^{-1}. \tag{2.1}$$

Definition 2.2. *Let $\bar{\mathcal{P}}(k, \mathbf{C}^q)$ denote the set of antiholomorphic polynomials on $\mathbf{C}^q$ that are homogeneous of degree k and let $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ denote the set of functions holomorphic on $\mathcal{D}_{p,q}$ taking values in $\bar{\mathcal{P}}(k, \mathbf{C}^q)$. Furthermore, for $\phi \in \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$, we put*

$$\phi(\zeta, v) := [\phi(\zeta)](v),$$

where $\zeta \in \mathcal{D}_{p,q}$ and $v \in \mathbf{C}^q$.

Definition 2.3. *For $k \geq 0$ we define $J_k : U(p,q) \times \mathcal{D}_{p,q} \to GL(\bar{\mathcal{P}}(k, \mathbf{C}^q))$ by*

$$J_k(g,\zeta)f(v) := \det[C\zeta + D]\ f\big([C\zeta + D]^* v\big)$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\zeta \in \mathcal{D}_{p,q}$ and $v \in \mathbf{C}^q$.

Definition 2.4. *Suppose $g \in G$ and $\phi \in \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$. Define $\omega_k(g)\phi$ by*

$$(\omega_k(g)\phi)(\zeta, v) = J_k(g^{-1}, \zeta)^{-1}\phi(g^{-1}.\zeta, v)$$

for $\zeta \in \mathcal{D}_{p,q}$ and $v \in \mathbf{C}^q$.

Observe that the representation ω_k is natural in the sense that it corresponds to the natural geometric action of G on sections of the vector bundle $\mathbf{E}_k = \mathcal{D}_{p,q} \times \bar{\mathcal{P}}(k, \mathbf{C}^q)$ via the factor of automorphy J_k (see [M]).

2.2 The Fock-disk Transform

In [M], a geometric construction of the positive spin ladder representations $\mathcal{F}_k$ is given via an integral transform Φ_k. We briefly explain the transform Φ_k and some of its properties. Further details may be found in [M].

Theorem 2.5. [M] *For $k \in \mathbf{Z}$, the mapping $\Phi_k : \mathcal{F}_k \to \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ given by*

$$(\Phi_k f)(\zeta, v) = \int_{\mathbf{C}^q} f(\zeta w, w) e^{v^* w} e^{-|w|^2}\ dm(w)$$

is well-defined and depends holomorphically on $\zeta \in \mathcal{D}_{p,q}$.

The transform Φ_k is essentially given by restriction to a negative q-plane followed by a Bargmann projection operator, giving rise to a kind of L^2-version of the Penrose transform. It should be pointed out that Φ_k is a trivialized version of the transform constructed in [M]. Specifically, we have used the global trivialization of the vector bundle in [M] with the multiplier action ω_k. Also, we have used the Fock space representation $\mathcal{F}_k$ instead of the L^2-cohomology realization of Blattner and Rawnsley (see [BR]).

Several properties of Φ_k will be particularly useful. We present the continuity and G-equivariant properties of Φ_k below.

Theorem 2.6. [M] *The mapping Φ_k is injective for $k \geq 0$ and identically zero for $k < 0$. Furthermore, for all $f \in \mathcal{F}_k$, $g \in G$, and $k \geq 0$, the mapping Φ_k satisfies*

$$\omega_k(g)(\Phi_k f) = \Phi_k(\sigma(g)f).$$

Lemma 2.7. [M] *Fix $k \in \mathbf{Z}$ and $\zeta \in \mathcal{D}_{p,q}$. The mapping of $\mathcal{F}_k$ into $\bar{\mathcal{P}}(k, \mathbf{C}^q)$ given by*

$$f \mapsto (\Phi_k f)(\zeta, \cdot)$$

is continuous.

3 Realizing ladder representations over G/H and the Penrose transform

In this section, we describe how the ladder representations $\mathcal{F}_k$ (section 2) for $G = \mathrm{SU}(p,q)$ may be realized in Dolbeault cohomology over a homogeneous space G/H of negative lines in $\mathbf{C}^{p+q}$ with coefficients in line bundles $E(-k-q)$. This is achieved by applying intertwining mappings Ψ_k constructed by Patton and Rossi (see [P] or [PR]). Along the way, we introduce a special case of the *Penrose transform*, denoted by P, which intertwines the Dolbeault cohomology realization of the ladder representations with those realized over G/K in section 3. This leads to the following commutative diagram.

$$\begin{array}{ccc} & \mathcal{F}_k & \\ \Psi_k \swarrow & & \searrow \Phi_k \\ H^{0,q-1}(G/H, E(-k-q)) & \xrightarrow{P} & \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q)) \end{array} \tag{3.1}$$

3.1 Preliminaries

Since we will be constructing cohomology from complexes of differential forms, we begin with some notation pertaining to our particular setting.

Definition 3.1. *For M a complex manifold and $\mathcal{V}$ a holomorphic vector bundle over M, let $\mathcal{E}^{r,s}(M,\mathcal{V})$ represent the space of* smooth $\mathcal{V}$-valued (r,s)-forms.

For our purposes, we will be working with $E(-k-q)$-valued (r,s)-forms on the complex manifold M_1, where M_1 consists of negative lines in $\mathbf{C}^{p+q}$ with respect to the indefinite metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle$, and $E(-k-q)$ is the $(k+q)$-th symmetric power of the tautological line bundle $E(-1)$ over M_1. Observe that G acts transitively on M_1, and that the line $[(0,\dots,0,1)]$ is stabilized by

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in G \mid A \in \mathrm{U}(p,q-1), d \in \mathrm{U}(1) \right\} \cong \mathrm{S}(\mathrm{U}(p,q-1) \times \mathrm{U}(1)).$$

Therefore M_1 is the homogeneous space G/H. Further, we note that $E(-k-q)$ is a G-homogeneous bundle, and the smooth sections of $E(-k-q)$ are $\mathbf{C}_\times$-homogeneous of degree $-(k+q)$.

Finally, we put

$$\mathbf{C}_-^{p+q} := \{ z \in \mathbf{C}^{p+q} \mid \langle\!\langle z,z \rangle\!\rangle < 0 \}, \tag{3.2}$$

and we let π denote the canonical mapping from $\mathbf{C}_-^{p+q}$ onto M_1.

3.2 $E(-k-q)$-valued forms and cohomology over G/H

We give the Dolbeault model for $H^r(M_1, \mathcal{O}(E(-k-q)))$ as sheaf cohomology, where $\mathcal{O}(E(-k-q))$ denotes the sheaf of holomorphic sections of $E(-k-q)$.

Then, using the natural mapping $\pi : \mathbf{C}_-^{p+q} \to M_1$, we pull Dolbeault cohomology back to cohomology over $\mathbf{C}_-^{p+q}$, into which the Patton-Rossi mappings Ψ_k will intertwine.

We first address the Dolbeault cohomology spaces over M_1. Put $n = p+q$. For $k \in \mathbf{Z}$ and $0 \leq r \leq n-1$, consider the space $\mathcal{E}^{0,r}(M_1, E(-k-q))$ of $E(-k-q)$-valued forms with associated $\bar{\partial}$-operator given by $\bar{\partial} = 1_{E(-k-q)} \otimes \bar{\partial}_{M_1}$. Here $\bar{\partial}_{M_1}$ is the del-bar operator for complex-valued forms on M_1, where the complex structure on M_1 is the one it inherits as an open G-orbit of $\mathbf{P}_{n-1}$. We then obtain a complex

$$0 \xrightarrow{i} \mathcal{E}^{0,0}(M_1, E(-k-q)) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{0,n-1}(M_1, E(-k-q)) \xrightarrow{\bar{\partial}} 0 \qquad (3.3)$$

which determines Dolbeault cohomology.

Definition 3.2. *For $k \in \mathbf{Z}$ and $0 \leq r \leq n-1$ we define the Dolbeault cohomology spaces by*

$$\mathcal{H}^{0,r}_{-k-q} = \frac{\operatorname{Ker}\{\bar{\partial} : \mathcal{E}^{0,r}(M_1, E(-k-q)) \to \mathcal{E}^{0,r+1}(M_1, E(-k-q))\}}{\operatorname{Im}\{\bar{\partial} : \mathcal{E}^{0,r-1}(M_1, E(-k-q)) \to \mathcal{E}^{0,r}(M_1, E(-k-q))\}},$$

where $\mathcal{E}^{0,-1}(\mathbf{P}_{n-1}, E(-k-q)) := 0$ and

$$\bar{\partial} : \mathcal{E}^{0,-1}(M_1, E(-k-q)) \to \mathcal{E}^{0,0}(M_1, E(-k-q))$$

is the inclusion mapping. For $r = n$, we put $\mathcal{H}^{0,n}_{-k-q} = 0$.

Following [PR], we proceed to pull the Dolbeault cohomology spaces described in Definition 3.2 back to $\mathbf{C}_-^{p+q}$. To begin, let

$$\Gamma := z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n} \qquad (3.4)$$

be the Euler vector field over $\mathbf{C}_-^{p+q}$. Since $T_{\pi(m)}(M_1) = T_m(\mathbf{C}_-^{p+q})/\mathbf{C}\Gamma$, pull-backs of forms in $\mathcal{E}^{r,s}(M_1, E(-k-q))$ to $\mathcal{E}^{r,s}(\mathbf{C}_-^{p+q})$ must vanish in the direction of the vector field Γ and must satisfy a homogeneity condition determined by the bundle $E(-k-q)$. Thus we define

$$\begin{aligned}\mathcal{E}^{r,s}_{-k-q} = \{\alpha \in \mathcal{E}^{r,s}(\mathbf{C}_-^{p+q}) \mid \iota(\Gamma)\alpha = \iota(\bar{\Gamma})\alpha = 0, \iota(\bar{\Gamma})d\alpha = 0,\\ \text{and } \iota(\Gamma)d\alpha = (-k-q)\alpha\},\end{aligned} \qquad (3.5)$$

where $\iota(\Gamma)$ denotes interior multiplication by Γ. We then produce a complex and resulting cohomology spaces over $\mathbf{C}_-^{p+q}$ (with complex structure inherited from $\mathbf{C}^{p+q}$) as follows.

Proposition 3.3. [PR] *The complex*

$$0 \xrightarrow{i} \mathcal{E}^{0,0}_{-k-q} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}_{-k-q} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{0,n}_{-k-q} \xrightarrow{\bar{\partial}} 0$$

is well-defined, and the corresponding cohomology spaces

$$H^{0,r}_{-k-q} = \frac{\mathrm{Ker}\{\bar{\partial} : \mathcal{E}^{0,r}_{-k-q} \to \mathcal{E}^{0,r+1}_{-k-q}\}}{\mathrm{Im}\{\bar{\partial} : \mathcal{E}^{0,r-1}_{-k-q} \to \mathcal{E}^{0,r}_{-k-q}\}}$$

satisfy $H^{0,r}_{-k-q} \cong \mathcal{H}^{0,r}_{-k-q}$ *for* $0 \leq r \leq (p+q)$.

3.3 Mapping into cohomology

Now that we have a suitable description of Dolbeault cohomology over $M_1 = G/H$, we are ready to present an intertwining mapping of $\mathcal{F}_k$ into cohomology following Patton and Rossi (see [P] or [PR]). Since the maximal dimension of compact subvarieties in M_1 is $q-1$, we expect the cohomology spaces $H^{0,r}_{-k-q}$ to be nonvanishing in degree $q-1$ (see [Z]). Hence we look to realize $\mathcal{F}_k$ within $H^{0,q-1}_{-k-q}$.

Definition 3.4. *Given* $f \in \mathcal{F}_k$ *and* $z \in \mathbf{C}^{p+q}_{-}$, *put*

$$\tilde{f}(z) = \int_{\mathbf{C}_\times} w^k |w|^{2(q-1)} f(wz) e^{-|z_S|^2|w|^2}\, dm(w).$$

Definition 3.5. *Define* $\Psi_k : \mathcal{F}_k \to \mathcal{E}^{0,q-1}(\mathbf{C}^{p+q}_{-})$ *by*

$$[\Psi_k(f)](z) = \frac{\tilde{f}(z)}{(k+q-1)!}\bar{\eta},$$

where

$$\eta = \sum_{j=1}^{q} (-1)^{j+1} z_{p+j}\, dz_{p+1} \wedge \cdots \wedge \widehat{dz}_{p+j} \wedge \cdots \wedge dz_{p+q}.$$

Theorem 3.6. [PR] *If* $k \geq 0$, *then* $\Psi_k(\mathcal{F}_k) \subset \mathcal{E}^{0,q-1}_{-k-q}$. *Furthermore, if* $\omega \in \Psi_k(\mathcal{F}_k)$, *then* $\bar{\partial}\omega = 0$, *and hence the mapping* $\Psi_k : \mathcal{F}_k \to \mathcal{E}^{0,q-1}_{-k-q}$ *induces a mapping* $[\Psi_k]$ *of* $\mathcal{F}_k$ *into the Dolbeault cohomology space* $H^{0,q-1}_{-k-q}$.

Note that we have not yet asserted that the mapping $[\Psi_k]$ *intertwines* the ladder representations into cohomology. This requires the Penrose transform, which will be addressed in the next subsection.

3.4 Penrose transform

We introduce a special case of the Penrose transform and see that the diagram given in (3.1) commutes. As a consequence, the mappings $[\Psi_k]$ intertwine the ladder repesentations $\mathcal{F}_k$ into cohomology. Readers desiring further knowledge of the Penrose transform may refer to [BE], [E], [EPW], [N], or [Z], to name a few.

In order to introduce the Penrose transform, we must first briefly discuss the mappings Ψ_k in the *compact case* $G = \mathrm{SU}(q)$ (that is, $p = 0$). In this case, $\mathcal{F}_k$ consists of antiholomorphic homogeneous polynomials on $\mathbf{C}^q$ of degree k (so that $\mathcal{F}_k = \bar{\mathcal{P}}(k, \mathbf{C}^q)$), and G acts irreducibly and unitarily on $\bar{\mathcal{P}}(k, \mathbf{C}^q)$ by left translation. Meanwhile, the cohomology space $\mathcal{H}^{0,q-1}_{-k-q}$ is finite dimensional over the compact, complex manifold $M_1 = \mathbf{P}_{q-1}$, and it "pulls back" to the cohomology space $H^{0,q-1}_{-k-q}$ over $\mathbf{C}^q_\times$ as described above in Proposition 3.3. By appealing to the Borel-Weil-Bott theorem, we know that $\bar{\mathcal{P}}(k, \mathbf{C}^q)$ and $H^{0,q-1}_{-k-q}$ are equivalent irreducible representations of $\mathrm{SU}(q)$. Mappings between these representations are induced by the mappings $\Psi_{k,c} : \bar{\mathcal{P}}(k, \mathbf{C}^q) \to \mathcal{E}^{0,q-1}_{-k-q}$ defined by

$$\Psi_{k,c} f(z) = \frac{f(z)}{|z|^{2(k+q)}} \bar{\eta}, \tag{3.6}$$

where η is as in Definition 3.5. In fact the $\Psi_{k,c}$ are exactly the mappings of Definition 3.5 in case $p = 0$. The following proposition is readily verified.

Proposition 3.7. (a) *For $k \geq 0$, the mapping $\Psi_{k,c} : \bar{\mathcal{P}}(k, \mathbf{C}^q) \to \mathcal{E}^{0,q-1}_{-k-q}$ is nonzero and* $\mathrm{SU}(q)$*-equivariant, where the action of* $\mathrm{SU}(q)$ *on $\mathcal{E}^{0,q-1}_{-k-q}$ is the natural action on forms induced by left translation.*

(b) *If $f \in \bar{\mathcal{P}}(k, \mathbf{C}^q)$, then $\Psi_{k,c}(f)$ is harmonic.*

The proposition indicates that $\Psi_{k,c}$ induces an intertwining mapping of $\mathcal{F}_k$ into $H^{0,q-1}_{-k-q}$, while the Hodge theorem together with the proposition imply that the induced mapping is nonzero. Thus we have:

Corollary 3.8. *The mapping $[\Psi_{k,c}] : \bar{\mathcal{P}}(k, \mathbf{C}^q) \to H^{0,q-1}_{-k-q}$ induced by $\Psi_{k,c}$ is a G-equivariant isomorphism.*

We are now ready to describe the Penrose transform in the setting $G = \mathrm{SU}(p, q)$ as described in [PR]. Let $[\omega]$ be a cohomology class in $H^{0,q-1}_{-k-q}$ represented by the form $\omega \in \mathcal{E}^{0,q-1}_{-k-q}$, let $\zeta \in \mathcal{D}_{p,q}$, and let $V(\zeta)$ be the negative q-plane in $\mathbf{C}^{p+q}$ spanned by the columns of $I(\zeta) = \begin{pmatrix} \zeta \\ I_q \end{pmatrix}$. Note that $V(\zeta) \cong \mathbf{C}^q$ as vector spaces via the coordinate mapping for $V(\zeta)$ with respect to the columns of $I(\zeta)$. Thus we may regard $\omega|_{V(\zeta)}$ as member of $\mathcal{E}^{0,q-1}(\mathbf{C}^q_\times)$, and in fact $\omega|_{V(\zeta)} \in \mathcal{E}^{0,q-1}_{-k-q}$. Now, by appealing to Corollary 3.8, we see that $[\Psi_{k,c}]^{-1}([\omega|_{V(\zeta)}]) \in \bar{\mathcal{P}}(k, \mathbf{C}^q)$. Thus we have associated a cohomology class $[\omega]$ with an element of $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ whose value at ζ is $[\Psi_{k,c}]^{-1}([\omega|_{V(\zeta)}])$. This mapping $H^{0,q-1}_{-k-q} \to \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ sending $[\omega] \mapsto [\Psi_{k,c}]^{-1}([\omega|_{V(\zeta)}])$ is the essence of the Penrose transform. However, to understand the geometry of the mapping, one must trace the definition through a double fibration of G/H and G/K, as described in [Z].

Theorem 3.9. [PR] *The mapping $P : H^{0,q-1}_{-k-q} \to \mathcal{E}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ given by*

$$(P[\omega])(\zeta) = [\Psi_{k,c}]^{-1}([\omega|_{V(\zeta)}])$$

for $[\omega] \in H^{0,q-1}_{-k-q}$ *and* $\zeta \in \mathcal{D}_{p,q}$ *is well-defined. Furthermore,* P *is injective,* P *is* G*-equivariant, and the diagram given in* (3.1) *commutes.*

Now, the fact that the Patton-Rossi mapping intertwines the ladder representations $\mathcal{F}_k$ into cohomology is an immediate corollary of Theorem 3.9:

Corollary 3.10. *For* $k \geq 0$*, the mapping* $[\Psi_k]$ *of* Theorem *3.6 induces a one-to-one,* G*-equivariant mapping from* $\mathcal{F}_k$ *into* $H^{0,q-1}_{-k-q}$*.*

4 Inverting the Penrose transform

We intertwine the realizations of ladder representations over G/K (section 2) into those in cohomology over G/H (section 3) in the case $k \geq 2(p+q)-1$ by explicitly constructing a G-equivariant mapping $P^{-1} : \Phi_k(\mathcal{F}_k) \to \Psi_k(\mathcal{F}_k)$. Furthermore, the mapping inverts the Penrose transform P in the sense that if $[\omega] \in \Psi_k(\mathcal{F}_k)$, then $[P^{-1} \circ P([\omega])] = [\omega]$. Roughly speaking, the mapping P^{-1} is constructed by passing through the Fock space $\mathcal{F}_k$, that is, we pull elements in $\Phi_k(\mathcal{F}_k)$ back to $\mathcal{F}_k$, and then send them into cohomology by using the mapping Ψ_k.

4.1 K-types and highest weight vectors

From Theorem 1.6 and Corollary 1.5, we observe that K acts on $\mathcal{F}_k$ by left translation. This action induces the following K-type decomposition.

Proposition 4.1. [SW] *For* $k \geq 0$*, the Fock space* $\mathcal{F}_k$ *decomposes into an orthogonal direct sum of* K*-types*

$$\mathcal{F}_k = \bigoplus_{s=0}^{\infty} (V^p_{-s} \otimes V^q_{k+s}),$$

where

$$V^p_{-s} = \{f(z_R) \mid f \in \mathcal{P}(s, \mathbf{C}^p)\}$$

and

$$V^q_{k+s} = \{f(z_S) \mid f \in \overline{\mathcal{P}}(k+s, \mathbf{C}^q)\}.$$

Here $\mathcal{P}(r, \mathbf{C}^m)$ *(resp.* $\bar{\mathcal{P}}(r, \mathbf{C}^m)$*) stands for holomorphic (resp. antiholomorphic) polynomials homogeneous of degree* r *on* $\mathbf{C}^m$*.*

With respect to the upper-triangular Borel subalgebra for $\mathfrak{k}_{\mathbf{C}} := \mathrm{Lie}(K)_{\mathbf{C}}$, highest weight vectors for the K-types $V^p_{-s} \otimes V^q_{k+s}$ are given by

$$f_{k,s} := z_p^s \bar{z}_{p+1}^{k+s}. \tag{4.1}$$

By applying the intertwining mappings Φ_k and Ψ_k to $f_{k,s}$ (see Theorem 2.5 and Definition 3.5), we obtain highest weight vectors for the corresponding K-types over G/K and G/H, respectively.

Proposition 4.2. *If $f_{k,s}$ is as in* (4.1)*, then $[\omega_{k,s}] := [\Psi_k(f_{k,s})]$ and $\phi_{k,s} := \Phi_k(f_{k,s})$ are $\mathfrak{k}_{\mathbf{C}}$ highest weight vectors in $\Psi_k(\mathcal{F}_k)$ and $\Phi_k(\mathcal{F}_k)$, respectively. Furthermore,*

$$\omega_{k,s}(z) = \frac{(k+s+q-1)!}{(k+q-1)!} \frac{z_p^k \bar{z}_{p+1}^{k+s}}{|z_{p+1}|^{2(k+s+q)}} \bar{\eta},$$

and

$$\phi_{k,s}(\zeta, v) = \frac{(k+s)!}{k!} \zeta_{p1}^s \bar{v}_1^k,$$

where $z \in \mathbf{C}_-^{p+q}$, $\zeta = (\zeta_{ij}) \in \mathcal{D}_{p,q}$, and $v \in \mathbf{C}^q$.

4.2 An orthogonal family of polynomials on G/K and norms of highest weight vectors

Here we lay the analytical groundwork on G/K that is needed for the construction of P^{-1}. We present a family of orthogonal polynomials on $\mathcal{D}_{p,q}$ constructed in [L1] and compute their norms with respect to certain K-invariant inner products. Further details may be found in [L1] and [L2].

We begin with notation.

Definition 4.3. (a) *For $\alpha \in \mathbf{N}_0^p$ and $\rho \in \mathbf{N}_0^q$, let*

$$M(\rho, \alpha) = \{\gamma \in \mathbf{N}_0^{p\times q} \mid c(\gamma) = \alpha \text{ and } r(\gamma) = \rho\},$$

where $c(\gamma)$ and $r(\gamma)$ denote the column and row sums of γ. (Note that $M(\rho, \alpha)$ is nonempty if and only if $|\alpha| = |\rho|$).

(b) *If $\gamma \in \mathbf{N}_0^{p\times q}$ and $\zeta = (\zeta_{ij}) \in \mathcal{D}_{p,q}$, we write*

$$\zeta^\gamma = \prod_{i=1}^p \prod_{j=1}^q \zeta_{ij}^{\gamma_{ij}} \text{ and } \gamma! = \prod_{i=1}^p \prod_{j=1}^q \gamma_{ij}!$$

Definition 4.4. *Fix $p, q, m \in \mathbf{N}$, and suppose $\alpha \in \mathbf{N}_0^p$ and $\rho \in \mathbf{N}_0^q$ with $|\alpha| = |\rho|$. For $\zeta \in \mathbf{C}^{p\times q}$, define*

$$\varphi_{\rho\alpha}(\zeta) := \sum_{\gamma \in M(\rho,\alpha)} \frac{1}{\gamma!} \zeta^\gamma,$$

and put

$$B_m := \{\varphi_{\rho\alpha} \mid \alpha \in \mathbf{N}_0^p(m), \rho \in \mathbf{N}_0^q(m)\}.$$

The polynomials $\varphi_{\rho\alpha}$ fit naturally into our framework in the following ways. First, the $\varphi_{\rho\alpha}$ are weight vectors for the natural action of $\mathfrak{k}_{\mathbf{C}}$ on the space $\mathcal{P}(\mathcal{D}_{p,q})$ of holomorphic polynomials on $\mathcal{D}_{p,q}$. Indeed, observe that the natural action of G on $\mathcal{D}_{p,q}$ given in (2.1) restricts to K as $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.\zeta = A\zeta D^{-1}$. This action induces an action of K on $\mathcal{P}(\mathcal{D}_{p,q})$, which in turn determines an action

$L \times R$ of $\mathfrak{gl}(p, \mathbf{C}) \oplus \mathfrak{gl}(q, \mathbf{C})$ on $\mathcal{P}(\mathcal{D}_{p,q})$. For $m \geq 0$, one may show that the polynomial ζ_{p1}^m is a highest weight vector for $L \times R$, and that B_m forms a basis of weight vectors for the irreducible $\mathfrak{gl}(p, \mathbf{C}) \oplus \mathfrak{gl}(q, \mathbf{C})$-module V_m with highest weight vector ζ_{p1}^m. In fact, one may obtain the polynomials in B_m by applying successive lowering operators to ζ_{p1}^m (see [L1]). Second, one may express elements in the image of the transform Φ_k in terms of the $\varphi_{\rho\alpha}$. Specifically, given a basis element $f_{\alpha\beta} = z_R^\alpha \bar{z}_S^\beta$ for $\mathcal{F}_k$, one computes

$$\Phi_k f_{\alpha\beta}(\zeta, v) = \sum_{\substack{\epsilon \in \mathbf{N}_0^q(k) \\ \epsilon \leq \beta}} \left(\alpha!\ \beta! \varphi_{(\beta-\epsilon),\alpha}(\zeta) \right) \frac{\bar{v}^\epsilon}{\epsilon!}, \tag{4.2}$$

where $\zeta \in \mathcal{D}_{p,q}$, $v \in \mathbf{C}^q$, and the order $\leq$ is as in Notation 1.1. Finally, we observe that the exponential function $\exp(\sum \zeta_{ij})$ on $\mathcal{D}_{p,q}$ may be expressed in terms of the $\varphi_{\rho\alpha}$.

Proposition 4.5. [L1] *The mapping of $\mathcal{D}_{p,q}$ into $\mathbf{C}$ given by*

$$\zeta \mapsto \exp\left(\sum_{i=1}^p \sum_{j=1}^q \zeta_{ij} \right)$$

has series expansion

$$\exp\left(\sum_{i=1}^p \sum_{j=1}^q \zeta_{ij} \right) = \sum_{m=0}^\infty \left(\sum_{\substack{\rho \in \mathbf{N}_0^q(m) \\ \alpha \in \mathbf{N}_0^p(m)}} \varphi_{\rho\alpha}(\zeta) \right).$$

We now discuss the norm and orthogonality properties of the polynomials $\varphi_{\alpha\beta}$.

Definition 4.6. *For $\lambda \geq (p+q)$, define an inner product $(\cdot,\cdot)_\lambda$ on $\mathcal{P}(\mathcal{D}_{p,q})$ by*

$$(f, g)_\lambda = \int_{\mathcal{D}_{p,q}} f(\zeta)\overline{g(\zeta)} d\mu_\lambda(\zeta),$$

where $d\mu_\lambda(\zeta) = \det(I_q - \zeta^*\zeta)^{[\lambda-(p+q)]} dm(\zeta)$.

Since K acts on $\mathcal{D}_{p,q}$ by linear automorphisms with determinant of modulus one, we see that the inner products $(\cdot,\cdot)_\lambda$ are K-invariant. Thus, any two distinct polynomials $\varphi_{\rho\alpha}$ and $\varphi_{\tilde{\rho}\tilde{\alpha}}$ must be orthogonal with respect to $(\cdot,\cdot)_\lambda$, as they are $\mathfrak{k}_{\mathbf{C}}$ weight vectors in $\mathcal{P}(\mathcal{D}_{p,q})$ with distinct weights. It remains to compute $(\varphi_{\rho\alpha}, \varphi_{\rho\alpha})_\lambda$. To do this, we first compute $(\zeta_{p1}^m, \zeta_{p1}^m)_\lambda$ using [H] and [FK], obtaining

$$(\zeta_{p1}^m, \zeta_{p1}^m)_\lambda = \frac{\Gamma_\Omega(\lambda - \frac{pq}{r})}{\Gamma_\Omega(\lambda)} . \frac{(\lambda-1)!m!}{(\lambda+m-1)!}, \tag{4.3}$$

If $\alpha = (0, \ldots, 0, m)$ and $\rho = (m, 0, \ldots, 0)$, then $\varphi_{\rho\alpha}(\zeta) = \frac{1}{m!}\zeta_{p1}^m$.

where $r = \min(p,q)$ and $\Gamma_\Omega(\lambda) = \prod_{j=1}^{r} \Gamma(\lambda - j + 1)$ is Gindikin's gamma function (see [FK]). Then, by using lowering operators, we may inductively compute $(\varphi_{\rho\alpha}, \varphi_{\rho\alpha})_\lambda$ for each $\varphi_{\rho\alpha} \in B_m$, giving the following theorem.

Theorem 4.7. [L1] *For $\lambda \geq (p+q)$, the collection of $\varphi_{\rho\alpha}$'s is orthogonal with respect to $(\cdot,\cdot)_\lambda$. Furthermore, if $m \geq 0$ and $\varphi_{\rho\alpha} \in B_m$ then*

$$(\varphi_{\rho\alpha}, \varphi_{\rho\alpha})_\lambda = \frac{\Gamma_\Omega(\lambda - \frac{pq}{r})}{\Gamma_\Omega(\lambda)} . \frac{(\lambda-1)!m!}{(\lambda+m-1)!} . \frac{1}{\alpha!\rho!}.$$

4.3 Intertwining mapping into cohomology

We construct an intertwining mapping from the disk realization of the ladder representations of G into Dolbeault cohomology. This mapping inverts the Penrose transform on its domain of definition.

Definition 4.8. *For $k \geq 2(p+q)-1$, $\phi \in \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$, and $z \in \mathbf{C}_-^{p+q}$, define $P^{-1}\phi \in \mathcal{E}^{0,q-1}(\mathbf{C}_-^{p+q})$ by*

$$(P^{-1}\phi)(z) = C \left[\int_{\mathbf{C}} \int_{\mathcal{D}_{p,q}} \phi(\zeta, z_S) |w|^{2(k+q-1)} e^{|w|^2 \langle\langle z, I(\zeta) z_S \rangle\rangle} d\mu_{k+1}(\zeta)\, dm(w) \right] \bar{\eta}$$

whenever the integral is defined, where $\zeta \in \mathcal{D}_{p,q}$, $I(\zeta) = \begin{pmatrix} \zeta \\ I_q \end{pmatrix}$, $\langle\langle\cdot,\cdot\rangle\rangle$ is the hermitian form of signature (p,q) on $\mathbf{C}^{p+q}$, $d\mu_{k+1}$ is as in Definition 4.6, η is as in Definition 3.5, and

$$C = \frac{1}{(k+q-1)!} . \frac{\Gamma_\Omega(k+1)}{\Gamma_\Omega(k+1-\frac{pq}{\min(p,q)})}.$$

Alternatively, observe that we express P^{-1} as an integral over G, giving

$$\begin{aligned}(P^{-1}\phi)(z) = C \Bigg[\int_{\mathbf{C}} \int_G & \frac{\phi(g, z_S)}{|\det g_{22}|^{2(k+1)}} \\ & \times |w|^{2(k+q-1)} e^{|w|^2 \langle\langle z, I(g.0) z_S \rangle\rangle} dg\, dm(w) \Bigg] \bar{\eta},\end{aligned} \tag{4.4}$$

where $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, and dg is an appropriate Haar measure on G.

In the next two lemmas, we show that P^{-1} is K-equivariant and has the desired inversion property on highest weight vectors for K-types.

Lemma 4.9. *Suppose ϕ is a K-finite vector in $\Phi_k(\mathcal{F}_k) \subset \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$. Then $P^{-1}\phi$ is defined, and given $g_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K$,*

$$P^{-1}(\omega_k(g_0)\phi) = g_0.(P^{-1}\phi),$$

where ω_k is the G-action on $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ given in Definition 2.4.

Proof. First, note that ϕ is K-finite if and only if ϕ is polynomial in $\zeta \in \mathcal{D}_{p,q}$, and hence the integral in Definition 4.8 converges, giving that $P^{-1}\phi$ is defined.

From Definition 2.4, we see that $[\omega_k(g_0)\phi](\zeta, v) = \det(\bar{D})\phi(g_0^{-1}.\zeta, D^{-1}v)$. Thus, for $z \in \mathbf{C}_-^{p+q}$, making the change of variable $\zeta \mapsto g_0.\zeta$ gives

$$\begin{aligned}
&P^{-1}(\omega_k(g_0)\phi)(z)\\
&\quad = \det(\bar{D})C\Bigg[\int_{\mathbf{C}}\int_{\mathcal{D}_{p,q}} \phi(g_0^{-1}.\zeta, D^{-1}z_S)|w|^{2(k+q-1)}e^{|w|^2\langle\langle z, I(\zeta)z_S\rangle\rangle}\\
&\qquad\qquad \times d\mu_{k+1}(\zeta)\, dm(w)\Bigg]\bar{\eta}\\
&\quad = C\Bigg[\int_{\mathbf{C}}\int_{\mathcal{D}_{p,q}} \phi(\zeta, D^{-1}z_S)|w|^{2(k+q-1)}e^{|w|^2\langle\langle z, I(g_0.\zeta)z_S\rangle\rangle}\\
&\qquad\qquad \times d\mu_{k+1}(\zeta)\, dm(w)\Bigg]\det(\bar{D})\bar{\eta}\\
&\quad = C\Bigg[\int_{\mathbf{C}}\int_{\mathcal{D}_{p,q}} \phi(\zeta, D^{-1}z_S)|w|^{2(k+q-1)}e^{|w|^2\langle\langle g_0^{-1}z, I(\zeta)(g_0^{-1}z)_S\rangle\rangle}\\
&\qquad\qquad \times d\mu_{k+1}(\zeta)\, dm(w)\Bigg](g_0.\bar{\eta})\\
&\quad = [g_0.P^{-1}\phi](z),
\end{aligned}$$

concluding the proof. ■

Lemma 4.10. *For $s \geq 0$ and $k \geq 2(p+q)-1$, we have*

$$P^{-1}\phi_{k,s} = \omega_{k,s},$$

where $\phi_{k,s}$ and $\omega_{k,s}$ are as in Proposition 4.2. Thus, by Proposition 4.2 and Theorem 3.9, $[P^{-1} \circ P[\omega_{k,s}]] = [\omega_{k,s}] \in H^{0,q-1}_{-k-q}$.

Proof. In succession, we apply Proposition 4.2, Proposition 4.5, Theorem 4.7, and Definitions 3.4 and 3.5 to obtain

$$\begin{aligned}
&P^{-1}\phi_{k,s}(z)\\
&\quad = C\frac{(k+s)!}{k!}\left[\int_{\mathbf{C}}\int_{\mathcal{D}_{p,q}} \zeta_{p1}^s \bar{z}_{p+1}^k |w|^{2(k+q-1)}e^{|w|^2\langle\langle z, I(\zeta)z_S\rangle\rangle} d\mu_{k+1}(\zeta)\, dm(w)\right]\bar{\eta}\\
&\quad = C\frac{(k+s)!}{k!}\Bigg[\int_{\mathbf{C}} |w|^{2(k+q-1)}\left(\int_{\mathcal{D}_{p,q}} \zeta_{p1}^s \bar{z}_{p+1}^k e^{|w|^2 z_R^t\overline{\zeta z_S}}\, d\mu_{k+1}(\zeta)\right)\\
&\qquad\qquad \times e^{-|w|^2|z_S|^2}\, dm(w)\Bigg]\bar{\eta}
\end{aligned}$$

$$= C\frac{(k+s)!}{k!s!} z_p^s \bar{z}_{p+1}^{k+s} \left[\int_{\mathbf{C}} |w|^{2(k+s+q-1)} \left(\int_{\mathcal{D}_{p,q}} |\zeta_{p1}|^{2s} \, d\mu_{k+1}(\zeta) \right) \right.$$

$$\left. \times \, e^{-|w|^2|z_S|^2} \, dm(w) \right] \bar{\eta}$$

$$= C\frac{(k+s)!}{k!s!} z_p^s \bar{z}_{p+1}^{k+s} \left[\int_{\mathbf{C}} |w|^{2(k+s+q-1)} \left(\frac{\Gamma_\Omega(k+1-\frac{pq}{r})}{\Gamma_\Omega(k+1)} \cdot \frac{k!s!}{(k+s)!} \right) \right.$$

$$\left. \times \, e^{-|w|^2|z_S|^2} \, dm(w) \right] \bar{\eta}$$

$$= \frac{1}{(k+q-1)!} \left[\int_{\mathbf{C}} w^k |w|^{2(q-1)} (wz_p)^s (\overline{wz_{p+1}})^{k+s} e^{-|w|^2|z_S|^2} \, dm(w) \right] \bar{\eta}$$

$$= [\Psi_k f_{k,s}](z)$$

$$= \omega_{k,s}(z),$$

which is the desired result. ■

Remarks 4.11. *At the beginning of the section, it was asserted that P^{-1} is given by first passing to the Fock space, and then sending the result into cohomology. This can be seen in the proof of Lemma 4.10. One observes in the second through the fifth displayed lines that the inner integral over $\mathcal{D}_{p,q}$ has the effect of sending the highest weight vector $\phi_{k,s} \in \Phi_k(\mathcal{F}_k)$ back to the corresponding highest weight vector $f_{k,s} \in \mathcal{F}_k$, using ideas from [DS] to eliminate limit expressions found in [L1] and [L2]. Then, the remaining integral over $\mathbf{C}$ is simply the Patton-Rossi mapping, which sends $f_{k,s}$ to cohomology.*

Since a K-finite vector $\phi \in \Phi_k(\mathcal{F}_k)$ may be obtained by applying $\mathbf{k_C}$-lowering operators to a finite collection of $\mathbf{k_C}$ highest weight vectors, it follows from Lemmas 4.9 and 4.10 that P^{-1} serves to invert the Penrose transform for K-finite vectors in $\Phi_k(\mathcal{F}_k)$. Of course, we have yet to show that P^{-1} is defined and possesses the desired inversion property for generic elements of $\Phi_k(\mathcal{F}_k)$. This will require a convergence argument, amid which we will need to invoke the Fubini theorem. The following lemmas, culminating in Lemma 4.14, will be needed for this purpose.

Lemma 4.12. *Suppose $k \geq 0$ and $\lambda \geq k+p+q+1$. Then*

$$\sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} \frac{|\alpha|!}{(|\alpha|+\lambda-1)!} \frac{|\beta|!}{(|\beta|-k)!} < \infty.$$

Proof. Let S denote the sum in the statement of the theorem. Using the fact that $\#(\mathbf{N}_0^r(s)) = \binom{r+s-1}{s}$, we have

$$S = \sum_{m=0}^{\infty} \sum_{\substack{\alpha\in\mathbf{N}_0^p(m)\\ \beta\in\mathbf{N}_0^q(m+k)}} \frac{(m+k)!}{(m+\lambda-1)!}$$

$$= \sum_{m=0}^{\infty} \#(\mathbf{N}_0^p(m))\#(\mathbf{N}_0^q(m+k))\frac{(m+k)!}{(m+\lambda-1)!}$$

$$= \frac{1}{(p-1)!(q-1)!}\sum_{m=0}^{\infty}\frac{(m+p-1)!}{m!}\frac{(m+k+q-1)!}{(m+\lambda-1)!}$$

$$\approx \sum_{m=0}^{\infty} m^{k+q+p-1-\lambda}.$$

We note that the last series above converges since $k+p+q-1-\lambda \le -2$, by hypothesis. ■

Lemma 4.13. *Suppose* $k+1 \ge p+q$, $\lambda = 2(k+1)-(p+q)$, *and* $\phi_{\alpha\beta} = \Phi_k(z_R^\alpha \bar{z}_S^\beta)$, *where* $|\beta|-|\alpha| = k$. *Then, for* $v \in \mathbf{C}^q$,

$$\int_{\mathcal{D}_{p,q}} |\phi_{\alpha\beta}(\zeta, v)|\ d\mu_{k+1}(\zeta) \le c\alpha!\beta!\left(\frac{|\alpha|!}{(|\alpha|+\lambda-1)!(|\beta|-k)!\alpha!}\right)^{\frac{1}{2}},$$

where c is a positive constant which does not depend on α or β.

Proof. For fixed $v \in \mathbf{C}^q$, we use (4.2) followed by the Schwarz inequality for integrals and Theorem 4.7 to obtain

$$\int_{\mathcal{D}_{p,q}} |\phi_{\alpha\beta}(\zeta,v)|\, d\mu_{k+1}(\zeta)$$

$$\le c_1 \int_{\mathcal{D}_{p,q}} \alpha!\beta! \left| \sum_{\substack{\epsilon\in\mathbf{N}_0^q(k)\\ \epsilon\le\beta}} \Big(\varphi_{(\beta-\epsilon),\alpha}(\zeta)\Big) \det(I_q - \zeta^*\zeta)^{k+1-p-q} \right| dm(\zeta)$$

$$\le c_2\left((\alpha!\beta!)^2 \int_{\mathcal{D}_{p,q}} \left| \sum_{\substack{\epsilon\in\mathbf{N}_0^q(k)\\ \epsilon\le\beta}} \Big(\varphi_{(\beta-\epsilon),\alpha}(\zeta)\Big) \det(I_q - \zeta^*\zeta)^{k+1-p-q} \right|^2 dm(\zeta)\right)^{\frac{1}{2}}$$

$$= c_2\alpha!\beta!\Bigg(\int_{\mathcal{D}_{p,q}} \sum_{\substack{\epsilon\in\mathbf{N}_0^q(k)\\ \epsilon\le\beta}} |\varphi_{(\beta-\epsilon),\alpha}(\zeta)|^2 \det(I_q-\zeta^*\zeta)^{2(k+1-p-q)}\, dm(\zeta)\Bigg)^{\frac{1}{2}}$$

$$= c_2\alpha!\beta!\Bigg(\int_{\mathcal{D}_{p,q}} \sum_{\substack{\epsilon\in\mathbf{N}_0^q(k)\\ \epsilon\le\beta}} |\varphi_{(\beta-\epsilon),\alpha}(\zeta)|^2 \, d\mu_{[2(k+1)-(p+q)]}(\zeta)\Bigg)^{\frac{1}{2}}$$

$$\le c_3\alpha!\beta!\Bigg(\sum_{\substack{\epsilon\in\mathbf{N}_0^q(k)\\ \epsilon\le\beta}} \frac{|\alpha|!}{(|\alpha|+\lambda-1)!(\beta-\epsilon)!\alpha!}\Bigg)^{\frac{1}{2}}$$

$$\le c_4\alpha!\beta!\Bigg(\frac{|\alpha|!}{(|\alpha|+\lambda-1)!(|\beta|-k)!\alpha!}\Bigg)^{\frac{1}{2}},$$

giving the desired inequality. ■

Lemma 4.14. *Suppose $k\ge 2(p+q)-1$ and $f\in\mathcal{F}_k$ with*

$$f(z)=\sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=k}} a_{\alpha\beta} z_R^{\alpha}\bar{z}_S^{\beta}.$$

Also, let $\phi_{\alpha\beta}$ be as in Lemma 4.13. Then

$$\sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=k}} \int_{\mathcal{D}_{p,q}} \left| a_{\alpha\beta}\phi_{\alpha\beta}(\zeta,z_S)e^{z_R^t\overline{\zeta z_S}}\right| d\mu_{k+1}(\zeta)<\infty.$$

Proof. Fix $z\in\mathbf{C}^{p+q}$. Using the fact that $e^{z_R^t\overline{\zeta z_S}}$ is bounded in modulus on $\mathcal{D}_{p,q}$ together with the Schwarz inequality for sums, we obtain

$$\sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=k}} \int_{\mathcal{D}_{p,q}} \left| a_{\alpha\beta}\phi_{\alpha\beta}(\zeta,z_S)e^{z_R^t\overline{\zeta z_S}}\right| d\mu_{k+1}(\zeta)$$

$$\le c_1\Bigg(\sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=k}} |a_{\alpha\beta}|^2\alpha!\beta!\Bigg)^{\frac{1}{2}}\Bigg(\sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=k}} \frac{1}{\alpha!\beta!}\Bigg(\int_{\mathcal{D}_{p,q}}|\phi_{\alpha\beta}(\zeta,z_S)|\,d\mu_{k+1}(\zeta)\Bigg)^2\Bigg)^{\frac{1}{2}}.$$

By Corollary 1.4, the lefthand factor appearing in the last displayed line is simply $\|f\|_{\mathcal{F}_k}$, which is finite. On the other hand, due to Lemmas 4.12 and 4.13, the righthand factor is finite when $k\ge 2(p+q)-1$. ■

Theorem 4.15. *Suppose that $k \geq 2(p+q)-1$ and $f \in \mathcal{F}_k$ with*

$$f(z) = \sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} a_{\alpha\beta} z_R^{\alpha} \bar{z}_S^{\beta}.$$

Further, let $\phi = \Phi_k f$ and $\omega = \Psi_k f$. Then, for $z \in \mathbf{C}_-^{p+q}$, $P^{-1}\phi(z) = \omega(z)$, and hence P^{-1} induces an inversion for the Penrose transform P on $\Phi_k(\mathcal{F}_k)$, where P^{-1} is the mapping defined in Definition 4.8.

Proof. For $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$ with $|\beta|-|\alpha| = k$, put $\phi_{\alpha\beta} = \Phi_k(a_{\alpha\beta} z_R^{\alpha} \bar{z}_S^{\beta})$ and $\omega_{\alpha\beta} = \Psi_k(a_{\alpha\beta} z_R^{\alpha} \bar{z}_S^{\beta})$. From Lemma 2.7 and Definition 3.5, we have

$$\phi(\zeta, v) = \sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} \phi_{\alpha\beta}(\zeta, v) \qquad \text{and} \qquad \omega(z) = \sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} \omega_{\alpha\beta}(z).$$

Since $\phi_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are K-finite vectors which, by Theorem 3.9, satisfy $P[\omega_{\alpha\beta}] = \phi_{\alpha\beta}$, Lemmas 4.9 and 4.10 imply that $P^{-1}(\phi_{\alpha\beta}) = \omega_{\alpha\beta}$.

Therefore, using (4.4) and Lemma 4.14 to invoke the Fubini theorem, we interchange integration and summation to obtain

$$P^{-1}\phi(z) = \left(\sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} P^{-1}\phi_{\alpha\beta}(z) \right) = \left(\sum_{\substack{\alpha \in \mathbf{N}_0^p \\ \beta \in \mathbf{N}_0^q \\ |\beta|-|\alpha|=k}} \omega_{\alpha\beta}(z) \right) = \omega(z),$$

concluding the proof. ■

4.4 The behavior of P^{-1} off the image of P; connections with certain differential operators

As a first step in considering an extension of P^{-1} to the entire image of the Penrose transform, we conclude by showing that P^{-1} annihilates K-finite vectors in $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ that are not in the image of P. In the process, we see that our construction of P^{-1} is intimately related to a set of differential operators which determine the image of P within $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$.

Recall that

$$P(H^{0,q-1}_{-k-q}) \subset \mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q)).$$

The K-finite elements of $\mathcal{O}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ are exactly those that have terminating power series in $\zeta \in \mathcal{D}_{p,q}$; thus

$$P(H^{0,q-1}_{-k-q})_K \subset \mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q)).$$

Furthermore, $P(H^{0,q-1}_{-k-q})_K$ is characterized within $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ by a set of differential operators. Specifically, we remark that $P(H^{0,q-1}_{-k-q})_K$ is the kernel of $\mathcal{X}$ in $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$, where

$$\mathcal{X} = \left\{ 2 \times 2 \text{ minors of } \begin{pmatrix} \frac{\partial}{\partial \zeta_{11}} & \frac{\partial}{\partial \zeta_{12}} & \cdots & \frac{\partial}{\partial \zeta_{1q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \zeta_{p1}} & \frac{\partial}{\partial \zeta_{p2}} & \cdots & \frac{\partial}{\partial \zeta_{pq}} \\ \frac{\partial}{\partial \bar{v}_1} & \frac{\partial}{\partial \bar{v}_2} & \cdots & \frac{\partial}{\partial \bar{v}_q} \end{pmatrix} \right\}, \tag{4.5}$$

with $(\zeta_{ij}) \in \mathcal{D}_{p,q}$ and $(v_1, \ldots, v_q) \in \mathbf{C}^q$.

Observe from Theorem 3.9, or from [DES], that these same operators characterize $\Phi_k(\mathcal{F}_k)_K$ as a subset of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$. In fact, there is an explicit multiplicity-free K-type decomposition of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ in which K-types of $\Phi_k(\mathcal{F}_k)$, whose highest weight vectors are described in Proposition 4.2, are exactly the K-types of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ that are annhiliated by the collection $\mathcal{X}$ of differential operators. Such a K-type decomposition may be obtained by first identifying $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ with $\mathcal{P}(\mathcal{D}_{p,q}) \otimes \bar{\mathcal{P}}(\mathbf{C}^q)$, then applying a theorem of Schmid in [S] to decompose $\mathcal{P}(\mathcal{D}_{p,q})$, and finally applying the Littlewood Richardson rule to the tensor product.

Thus, from the discussion immediately following Definition 4.4, along with Proposition 4.5, we see that the exponential kernel that appears in the definition of P^{-1} (see Definition 4.8) is essentially compiled from weight vectors of these distinguished K-types for $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$, and as such is naturally designed to detect elements of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ that are killed by $\mathcal{X}$, i.e., to detect elements in $\mathcal{P}(H^{0,q-1}_{-k-q})_K$. Thus we expect that P^{-1} will annhiliate any element of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ that lies in the direct sum of the remaining K-types (that is, K-types that are not killed by $\mathcal{X}$).

So, let V denote the direct sum of K-types of $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q))$ that annhiliated by $\mathcal{X}$, while letting W denoted the direct sum of the K-types that are not killed by $\mathcal{X}$. Then $\mathcal{P}(G/K, \bar{\mathcal{P}}(k, \mathbf{C}^q)) = V + W$, and we have the following proposition.

Proposition 4.16. *Suppose $\phi(\zeta, v) \in W$. Then $P^{-1}\phi = 0$.*

Proof. By linearity, we may assume that ϕ lies within a single K-type, denoted by M. Since $\phi(\zeta, v)$ is polynomial in ζ, $P^{-1}\phi$ will converge.

From the proof of Lemma 4.10, we see that for $z \in \mathbf{C}^{p+q}_-$,

$$\begin{aligned} &P^{-1}\phi(z) \\ &\quad = C\left[\int_{\mathbf{C}} |w|^{2(k+q-1)} \left(\int_{\mathcal{D}_{p,q}} \phi(\zeta, \bar{z}_S) e^{|w|^2 z_R^t \overline{\zeta z_S}} \, d\mu_{k+1}(\zeta) \right) \right. \\ &\qquad\qquad \left. \times e^{-|w|^2 |z_S|^2} \, dm(w) \right] \bar{\eta}, \end{aligned}$$

and thus it suffices to show that

$$\int_{\mathcal{D}_{p,q}} \phi(\zeta, \bar{z}_S) e^{z_R^t \overline{\zeta z_S}}\, d\mu_{k+1}(\zeta) = 0.$$

Now, observe that the mapping from M to $\mathcal{F}_k$ given by

$$\psi(\zeta, v) \mapsto \int_{\mathcal{D}_{p,q}} \psi(\zeta, \bar{z}_S) e^{z_R^t \overline{\zeta z_S}}\, d\mu_{k+1}(\zeta) \tag{4.6}$$

is well defined and K-equivariant, and so the mapping is either identically zero or it gives an isomorphism of M onto a K-type within $\mathcal{F}_k$. Since the K-types of $\mathcal{F}_k$ are isomorphic to those in V via Φ_k, and M is not isomorphic to any element of V, we conclude that the mapping (4.6) is identically zero. Therefore $P^{-1}\phi = 0$ as claimed. ■

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