Review problems for Exam 1.

1. Let R be a commutative ring with $1 \neq 0$. Prove that the ideal (x) is prime in $R[x]$ iff R is an integral domain. Prove that (x) is maximal ideal iff R is a field.

2. Let R be a commutative ring with $1 \neq 0$. Prove that if a is a nilpotent element of R then $1 - ab$ is a unit for all $b \in R$.

3. Prove that if R and S are nonzero rings then $R \times S$ is never a field.

4. Let R be a ring for which every ideal is finitely generated. Prove that there is no infinite strictly increasing sequence of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots \subset R$.

5. Prove that in a PID two ideals (a) and (b) are comaximal iff a greatest common divisor of a and b is 1.

6. Prove that a quotient of a PID by a prime ideal is again a PID.

7. Prove that the ideals (x) and (x, y) are prime ideals in $\mathbf{Q}[x, y]$ and the second one is maximal.

8. Let F be a finite field of order q and let $f(x)$ be a polynomial in $F[x]$ of degree $n \geq 1$. Prove that $F[x]/(f(x))$ has q^n elements.

9. Find all the ideals in the ring $F[x]/(f(x))$, where F is a field, in terms of the factorization of $f(x)$ into irreducibles.