

# Power Putnam Preparation

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## 1 About the Putnam Exam

The Putnam Exam is a challenging examination in mathematical problem-solving that has been given each year in the U.S. and Canada since 1938. It is open to any undergraduate. A student can take the exam at most four times. It consists of two three-hour sessions in each of which six problems are given to be solved. The problems are very difficult, and solving even one or two is a fine accomplishment. Generally, the problems all have brief but ingenious solutions. Currently, more than 2000 students take the exam each year.

The winners receive substantial prizes. The top prize is a one-year fellowship to Harvard equal to tuition plus \$12,000. The top score in the Oklahoma-Arkansas section of the MAA also receives a prize, and the top score at OSU is memorialized on a plaque in the Math Department. A list of the top 500 finishers is prepared and circulated to universities around the country. Finishing in the top 500 is a distinction that attracts the attention of most graduate departments of mathematics.

Old exams are published (along with the results) each year in the *American Math. Monthly*. There are several excellent references on Putnam problems and problem-solving in general. See the bibliography at the end.

## 2 Related Activities

There are “problems columns” in many math journals, including:

- *American Mathematical Monthly*
- *Mathematics Magazine*
- *The College Mathematics Journal*

It is excellent preparation to work on these problems. Moreover, if you solve them and write up your solution, you can submit it for publication in these journals.

Recently, the **Courant Institute in New York** has instituted an essay contest in mathematics. Often Putnam problems have extremely interesting generalizations or related problems. You may submit an original solution of a challenging Putnam problem, or preferably a generalization, to this contest.

### 3 Subject matter of the exam

The Putnam exam is based mainly on the beginning subjects of an undergraduate mathematics major. Here is a sampling of subjects that have appeared on past exams:

**Polynomials:** Factoring, Fundamental theorem of algebra

**Set Theory:** Venn diagrams, unions and intersections, countability, one-to-one correspondences

**Combinatorics:** Choosing subsets of sets, pigeonhole principle, graphs and coloring

**Recurrence:** Solution of linear recurrence relations, induction proofs

**Linear Algebra:** Independence, bases, matrices, invertibility, determinants (Vandermonde in particular), block decomposition

**Number Theory:** Prime factorization, congruences, Fermat's little theorem, quadratic reciprocity, generating functions, approximating irrational numbers by rationals

**Binomials:** Pascal's triangle, expansion of  $(x + y)^n$ .

**Abstract Algebra:** Ring theory, isomorphisms

**Field Theory:** Finite fields

**Group Theory:** Basic group theory, Lagrange's theorem, characters, orthogonality, representations

**Geometry:** analytic geometry, conic sections, inequalities, circles, triangles (special points: incenter, circumcenter, centroid)

**Inequalities:** Triangle inequality, Cauchy-Schwarz, inequality of arithmetic and geometric means

**Limits–Continuity:** Squeezing theorem

**Functional Equations:** Iterating functions

**Extremal Problems:** Critical points, gradient

**Differential Calculus:** Chain rule in several variables

**Integral Calculus:** substitution, partial fractions, special functions

**Probability:** Probabilities arising as integrals or areas, expectation

**Series:** Convergence tests, rearrangement of series, geometric series, binomial series

**Vector Calculus:** Stokes' theorem, Green's theorem, potentials

**Differential Equations:** Linear equations, constant coefficient equations, Laplace equation

## 4 A Sample Problem

Here is a sample problem and write-up of solution. You must include all the exposition necessary to justify your solution. You must write clearly, using full sentences which are grammatically correct.

**Problem:** Find, with explanation, the maximum value of  $f(x) = x^3 - 3x$  on the set of all real numbers  $x$  satisfying  $x^4 + 36 \leq 13x^2$ .

**Solution:** The maximum value is 18. (Always state the answer at the beginning of your write-up.)

**Proof:** First, we show that  $x^4 + 36 \leq 13x^2$  if and only if  $-3 \leq x \leq -2$  or  $2 \leq x \leq 3$ .

$$\begin{aligned}x^4 + 36 \leq 13x^2 &\Leftrightarrow x^4 - 13x^2 + 36 \leq 0 \\&\Leftrightarrow (x^2 - 4)(x^2 - 9) \leq 0 \text{ by factoring} \\&\Leftrightarrow 4 \leq x^2 \leq 9 \\&\Leftrightarrow 2 \leq |x| \leq 3\end{aligned}$$

This proves our first claim.

The critical points of  $f(x)$  occur when  $f'(x) = 3x^2 - 3 = 0$ , namely, when  $x = \pm 1$ . Since neither critical point lies in  $[-3, -2]$  or  $[2, 3]$ , the extremal values of  $f(x)$  in these intervals must occur at the endpoints. Then we check  $f(-3) = -18$ ,  $f(-2) = -2$ ,  $f(2) = 2$ , and  $f(3) = 18$ . This proves our answer is correct. **QED**

You are allowed abbreviations like  $\Leftrightarrow$  for “if and only if.” However, each key point in the proof must be presented clearly and in an orderly fashion. Each problem is graded on a 10 point scale; however, partial credit is very difficult to obtain on the Putnam Exam.

## 5 Preparation and Strategy

All human accomplishment is based on strength of will and sound preparation. The Putnam Exam does not belong to the “super-geniuses;” it is possible to improve your skills at analyzing and solving these problems to any given level. The phenomenal success that Olympiad

training camps have had at doing just this prove this beyond doubt. It's simply a question of making that kind of training available to every student.

### Basic tenets

- The more mathematics you know, the better you'll do.
- Most Putnam problems are based in classic techniques and problems from mainstream mathematics.
- Emphasize the theorems used; categorize the problems according to the branch of mathematics they come from. This classification helps you think of the relevant techniques to use.

## 5.1 Plan of Attack

It is useful to have a carefully laid plan of attack in studying the problems on the exam. Here are some suggestions.

1. Start by reading all six problems completely through. **Do not write anything until you have done this!**
2. Read the problems through again and make notes on the six problems:
  - (a) What category of math: calculus, linear algebra, number theory, etc.
  - (b) Underline the **data** in each problem
  - (c) Estimate the difficulty and devise an order of attack. While A-1 and B-1 are usually the easiest in each session, that may not be the case for a given year and a given student.
3. Start to work on what you think is the easiest problem. **Work only on scratch paper first.**
4. Allocate at most 30 minutes to work on any given problem. If at the end of that 30 minutes you haven't solved it, move on to the next problem. If you do solve it, take the time to carefully transfer your work to the solution folder. The write-up is extremely important; if you miss key points, you will receive little credit.
5. In the last fifteen minutes of the session, transfer whatever partial work you have done to the folders in hopes of gaining partial credit. State the answers clearly; you may gain some credit for a correct answer without a correct proof.
6. Be sure that all solution folders and extra pieces of paper that you are turning in have **an ID sticker** on them.

## 5.2 Working on Individual Problems

1. Sometimes it helps to write the problem word-for-word on scratch paper. Underline all important data and phrases.
2. Write down statements of all relevant facts and theorems you can think of.
3. Try to break the problem into easier subproblems.
4. Try to reduce the number of dimensions, number of variables, etc., to see if you can solve an easier version of the problem. That may give you a clue how to solve the original problem.
5. Draw a picture. Draw and redraw until your picture matches all the data of the problem.
6. Introduce notation for all the quantities described in the problem. Be very careful to use different notation for possibly different quantities. Get comfortable with using subscripts and double-subscripts for describing sets of variables.
7. State all possible relations between the quantities stated in the problems.
8. Frequently reread the problem to make sure you have no misunderstanding.
9. To save time, never erase on scratch paper. Write large and leave plenty of space to allow corrections. For long formulas, arrange the paper with the long side horizontal. If you need to cross out a lot, simply start again on a fresh sheet of paper and copy the good part of your work.
10. Check to make sure you use all the hypotheses of your problem.

## References

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