THE QUADRIC SURFACES

Suppose we have a general quadratic equation in three variables:

$$A\,x^2 + B\,y^2 + C\,z^2 + D\,x\,y + E\,y\,z + F\,x\,z + G\,x + H\,y + I\,z + J = 0$$

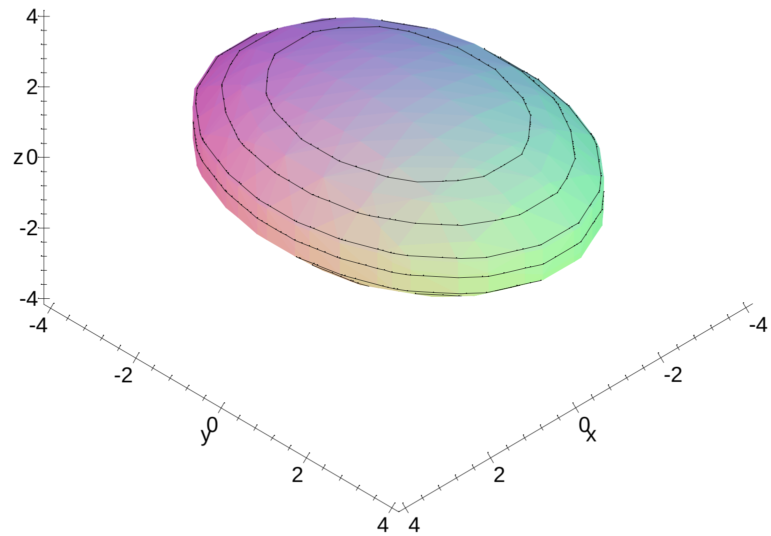
It can be shown that by using translations and rotations of space to change variables one can rewrite the equation so that it either has one of the nine forms listed below (the quadric surfaces) or is "degenerate" (for example, a point, the empty set, a pair of planes, etc.). In what follows a **contour** will be the intersection of the graph with a horizontal plane, and a **section** will be the intersection of the graph with a vertical plane. Note that by permuting the variables you can get graphs with different orientations, and that the roles of some contours and sections may be interchanged.

1. The **ellipsoid** has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The contours and sections are ellipses, points, and the empty set.
There are six vertices: (a,0,0), (-a,0,0), (0,b,0), (0,-b,0), (0,0,c), and (0,0,-c).

Here is the example $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$:

```
> with(plots):
> implicitplot3d(x^2/9+y^2/16+z^2/4=1,x=-4..4,y=-4..4,z=-4..4,axes=frame,style=patchcontour);
```

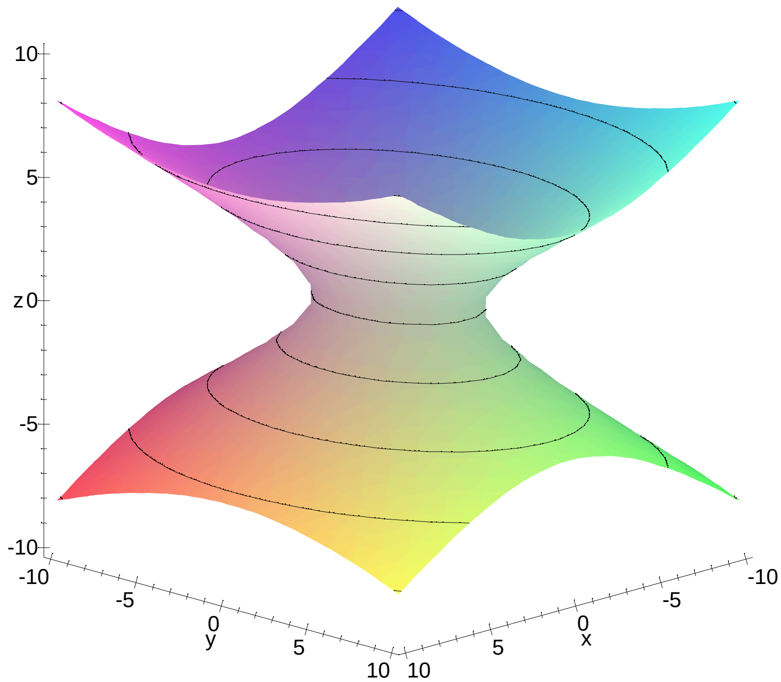


2. The **hyperboloid of one sheet** has equation $\frac{x^2}{z^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

The contours are ellipses. The sections are hyperbolas and pairs of crossed lines.

Here is the example $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{4} = 1$:

```
> implicitplot3d(x^2/9+y^2/16-z^2/4=1,x=-10..10,y=-10..10,z=-10..10,
  axes=frame,style=patchcontour,orientation=[45,75],grid=[20,20,20])
  ;
```

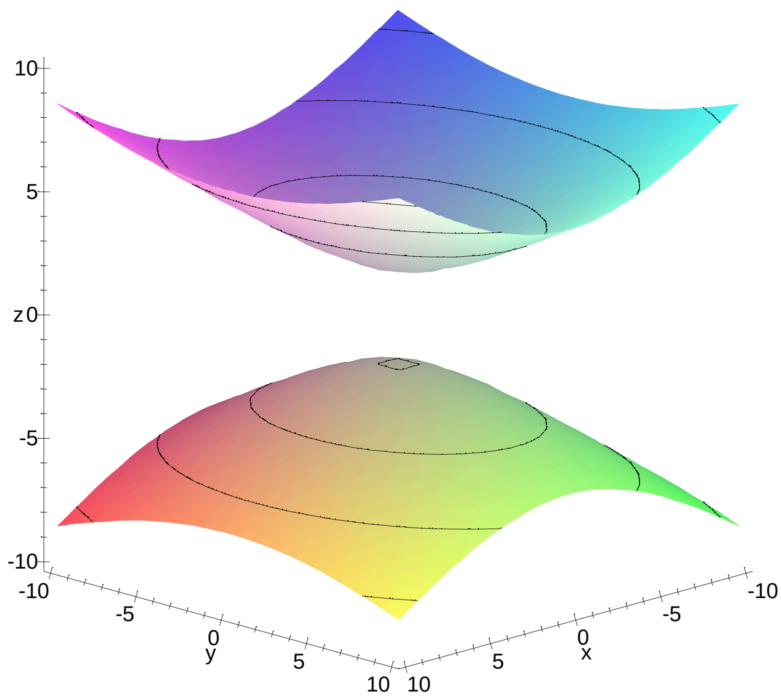


3. The **hyperboloid of two sheets** has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$.

The contours are ellipses, points, and the empty set. The sections are hyperbolas.
There are two vertices: (0,0,c) and (0,0,-c)

Here is the example $\frac{x^2}{9}+\frac{y^2}{16}-\frac{z^2}{4}=-1$:

```
> implicitplot3d(x^2/9+y^2/16-z^2/4=-1,x=-10..10,y=-10..10,z=-10..10
  ,axes=frame,style=patchcontour,orientation=[45,75],grid=[20,20,20]
  );
```

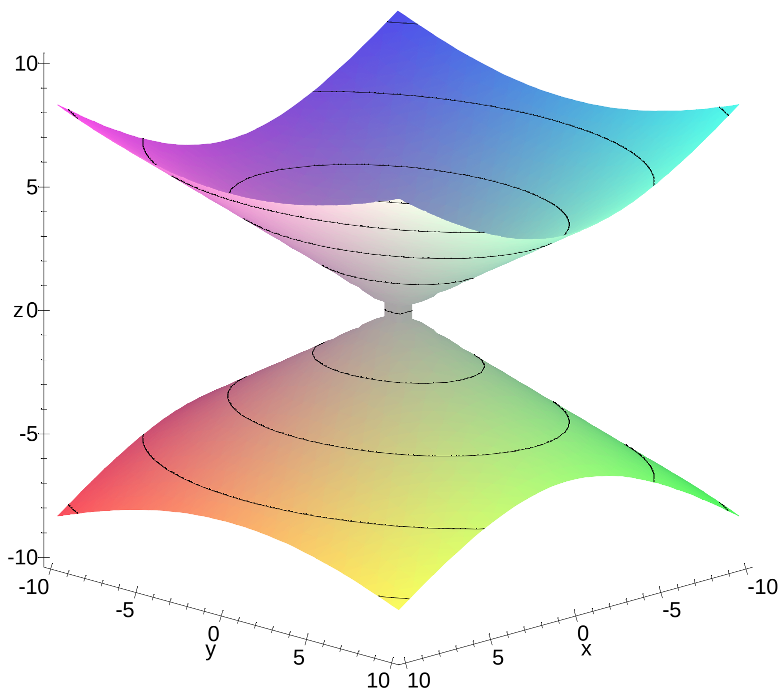


4. The **quadric cone** has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$.

The contours are ellipses and a point. The sections are hyperbolas and pairs of crossed lines.
There is one singular point: (0,0,0)

Here is the example $\frac{x^2}{9}+\frac{y^2}{16}-\frac{z^2}{4}=0$:

```
> implicitplot3d(x^2/9+y^2/16-z^2/4=0,x=-10..10,y=-10..10,z=-10..10,
  axes=frame,style=patchcontour,orientation=[45,75],grid=[30,30,30])
  ;
```

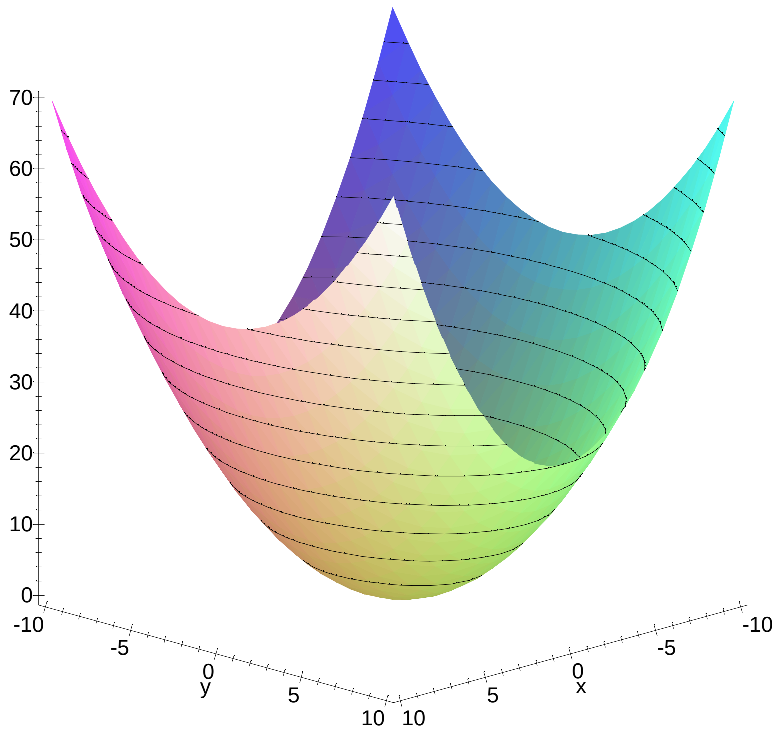


5. The **elliptic paraboloid** has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$.

The contours are ellipses, a point, and the empty set. The sections are parabolas.
(This is how to remember the name.)
There is one vertex: (0,0,0)

Here is the example $\frac{x^2}{9}+\frac{y^2}{16}=\frac{z}{4}$:

```
> plot3d(4*(x^2/9+y^2/16),x=-10..10,y=-10..10,axes=frame,style=patch
  contour,orientation=[45,75]);
```

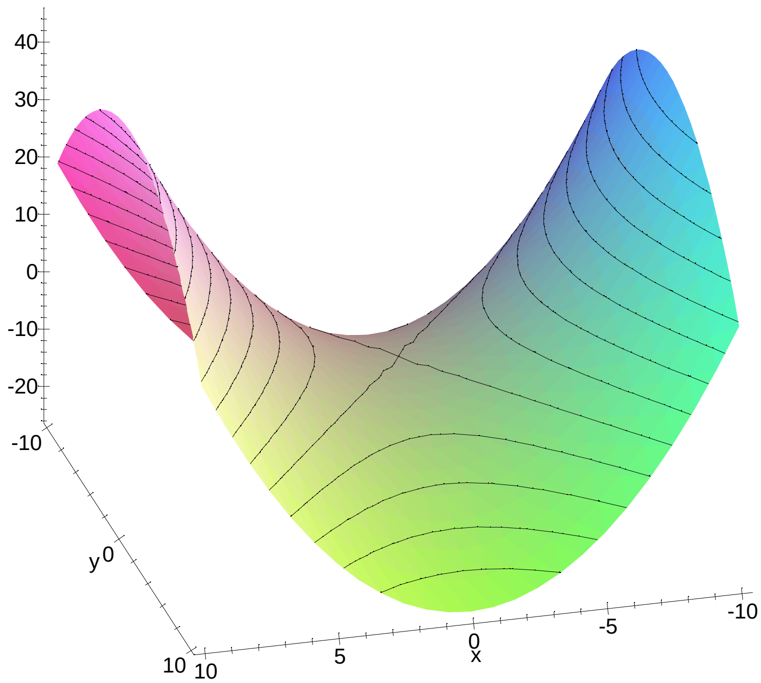


6. The **hyperbolic paraboloid** has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$.

The contours are hyperbolas and a pair of crossed lines. The sections are parabolas and lines. (Again, this is how to remember the name.)

Here is the example $\dfrac{x^2}{9} - \dfrac{y^2}{16} = \dfrac{z}{4}$:

```
> plot3d(4*(x^2/9-y^2/16),x=-10..10,y=-10..10,axes=frame,style=patch
  contour,orientation=[75,60]);
```

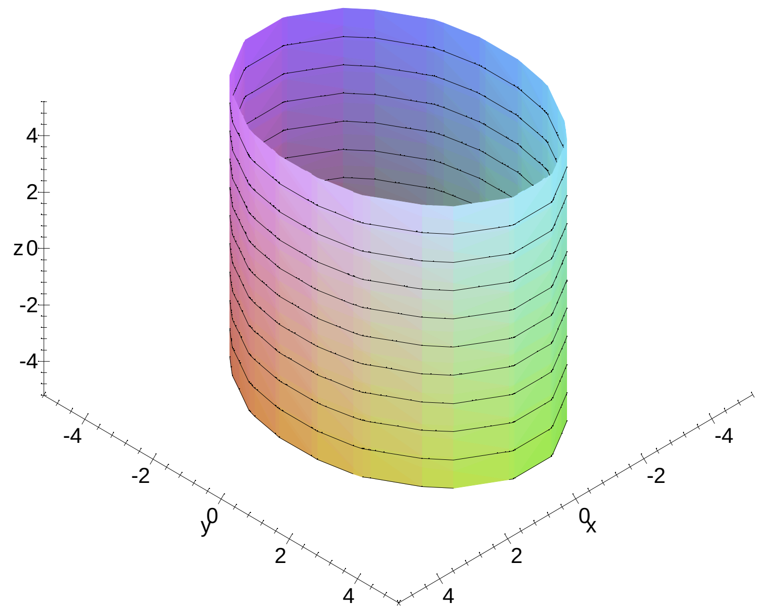


7. The **elliptic cylinder** has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The contours are ellipses. The sections are pairs of parallel lines, lines, and the empty set.

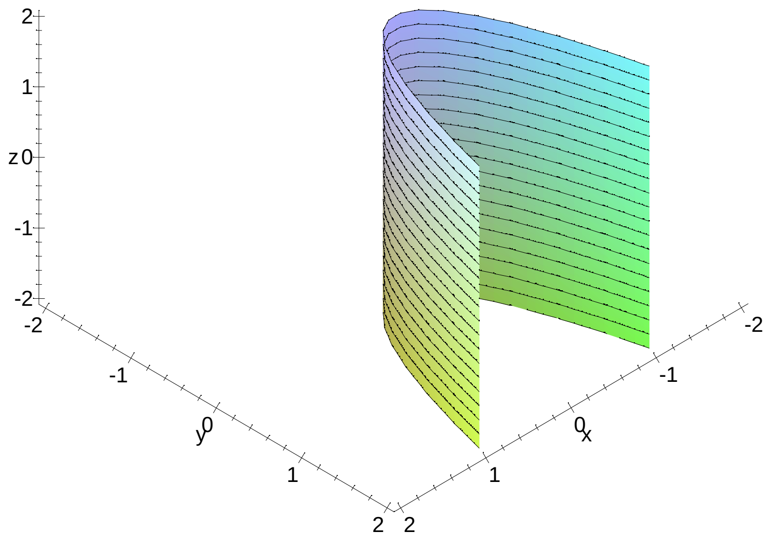
Here is the example $\frac{x^2}{9} + \frac{y^2}{16} = 1$:

```
> implicitplot3d(x^2/9+y^2/16=1,x=-5..5,y=-5..5,z=-5..5,axes=frame,style=patchcontour);
```



8. The **parabolic cylinder** has equation $y = c\,x^2$.
The contours are parabolas. The sections are pairs of parallel lines, a line, and the empty set.
Here is the example $y = 2\,x^2$:

```
> implicitplot3d(y=2*x^2,x=-2..2,y=-2..2,z=-2..2,axes=frame,style=pa
  tchcontour,grid=[30,30,30]);
```

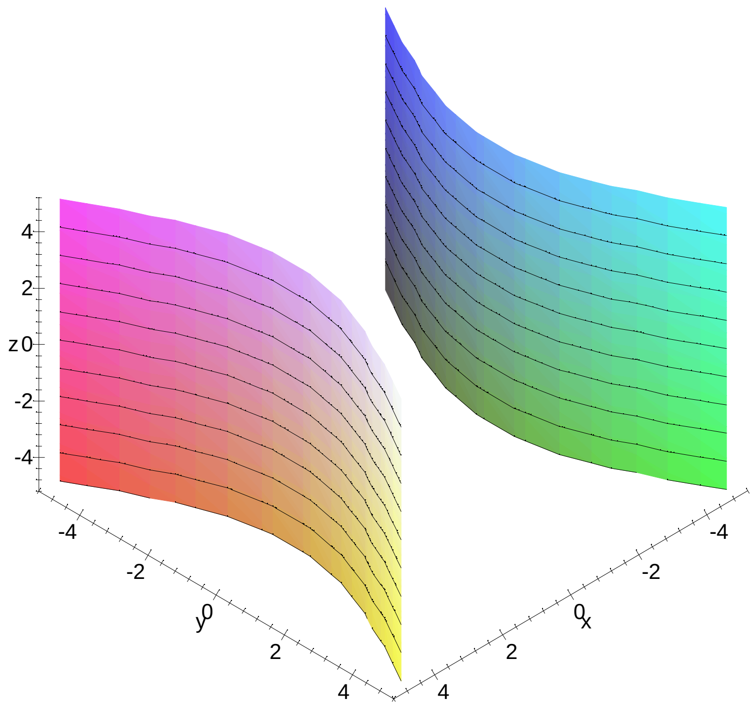


9. The **hyperbolic cylinder** has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The contours are hyperbolas. The sections are pairs of parallel lines, lines, and the empty set.

Here is the example $\frac{x^2}{9} - \frac{y^2}{16} = 1$:

```
> implicitplot3d(x^2/9-y^2/16=1,x=-5..5,y=-5..5,z=-5..5,axes=frame,s
  tyle=patchcontour);
```



4
2
z 0
-2
-4
-4
-2
y 0
2
4
4
2
0 x
-2
-4