Team Number:

1. How many different ordered pairs (x, y) of positive integers satisfy the equation $3x + 7y = 100$?

Solution: We have $3(-2)+7(1) = 1$, so $3(-2000)+7(1000) = 1000$. Since $\gcd(3, 7) = 1$, the solutions to $3a + 7b = 0$ all have the form $(a, b) = (7k, 3k)$ for an integer k.

Thus the solutions to $3x+7y = 1000$ all have the form $(x, y) = (-2000+7k, 1000-3k)$ for some k. We thus need k to be an integer between $\frac{2000}{7}$ and 10003, i.e., $k \in \{286, 333\}$. There are 48 solutions.

Team Number:

2. Frank and Jeremy are playing a card game to determine the champion of the world. They agree to play a series of five games; whoever wins three or more will be the champion. If the probability of Jeremy winning each game is given by $\frac{2+5f-3f^2}{4}$, where f is the number of games that Frank has already won, what is the probability that Jeremy will become the champion?

Solution: If Frank wins a game, the probability of his winning another game becomes $1 - \frac{2+5-3}{4} = 0$. Thus Jeremy wins all remaining games and becomes champion with probability 1.

Team Number:

3. A high school math class with 20 students is walking down a hallway when they see 20 closed, but unlocked, lockers along the wall, numbered 1 through 20. The first student to go down the hall opens all of the lockers. The second wants to be different, and so they go down and close the even numbered lockers $2, 4, 6, \ldots, 20$, but leave the odd numbered lockers alone. The third student notices a pattern is forming, so they decide that they want to follow it, and they go along and flip the lockers that are multiples of 3, that is, $3, 6, 9, 12, \ldots, 18$, in other words, they open the locker if it was closed, but close it if it was open. They continue in this pattern: the fourth student flips lockers $4, 8, 12, \ldots$, that is, all the multiples of 4, the fifth student all multiples of 5, etc. Which lockers are left open after the math class is finished having their fun?

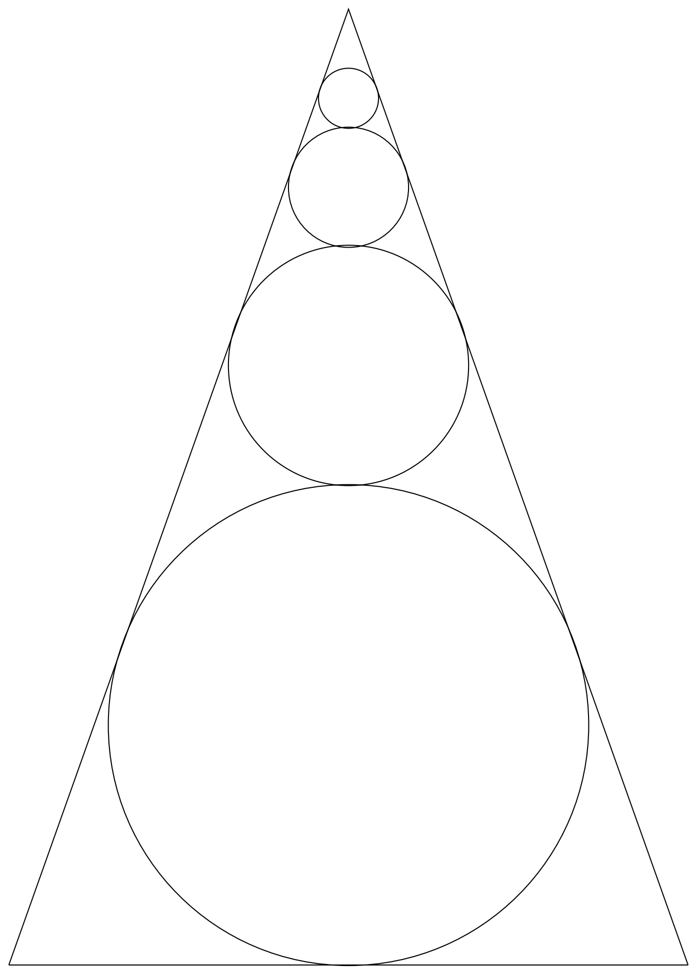
Solution: Locker number n is flipped by student number k if and only if k is a factor of n. So locker n is flipped $d(n)$ times, where $d(n)$ is the number of factors of n. We conclude that locker n is left open if and only if $d(n)$ is odd.

The factors of n can be organized into pairs $(k, \frac{n}{k})$. The only way that $d(n)$ can be odd is if one of these pairs consists of only one number, i.e., $k = \frac{n}{k}$, i.e., $n = k^2$.

So locker n is left open if and only if n is a perfect square. That is, lockers 1, 4, 9, and 16 are left open and all others are left closed.

Team Number:

4. In the figure, an infinite tower of circles is inscribed in an isosceles triangle. Each circle is externally tangent to the one above it, and each has twice the radius of the one above it. If the bottom circle has radius 1 and the base of the triangle has length $2\sqrt{2}$, find the triangle's area.



Solution: The triangle's height is the sum of the diameters of the circles. These form a geometric sequence with first term 2 and common ratio $\frac{1}{2}$. Thus we have $h = \dfrac{2}{1-\frac{1}{2}} = 4.$

The area is $\frac{1}{2}\left(2\sqrt{2}\right)\left(4\right) = 4\sqrt{2}.$

Team Number:

5. Suppose that $x = 2$ is a root of the cubic polynomial $x^3 + 3x^2 + bx + 6$. Find b.

Solution: We have $2^3 + 3(4) + b(2) + 6 = 0$, so $b = -13$.