LECTURE 15

Identities of Vector Analysis

1. Differential Operator Notation

Let ∇ denote the formal symbol

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

thought of as a 3-dimensional vector. Of course, the components of ∇ really don't make any sense until they act of a function. But if we permit ourselves this notational absurdity, we can better understand the notation used for the gradient, divergence and curl:

$$\begin{aligned}
\nabla f &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\\
\nabla\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(F_x,F_y,F_z) = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\\
\nabla\times\mathbf{F} &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(F_x,F_y,F_z)\\
&= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z},\frac{\partial F_x}{\partial z}-\frac{\partial F_x}{\partial z},\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)
\end{aligned}$$

Along these same lines we now introduce a second order differential operator, the **Laplacian** that is defined by

$$\nabla^2 f \equiv (\nabla\cdot\nabla)f = \nabla\cdot(\nabla f) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

THEOREM 15.1. *(Fundamental Identities of Vector Analysis). Let f and g be real-valued functions on $\mathbb{R}^3$ and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields on $\mathbb{R}^3$. Then*

1. $\nabla(f+g) = \nabla f+\nabla g$
2. $\nabla(cf) = c(\nabla f)$ *for any constant* c
3. $\nabla(fg) = g(\nabla f)+f(\nabla g)$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$
5. $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F}+\nabla\cdot\mathbf{G}$
6. $\nabla\times(\mathbf{F}+\mathbf{G}) = \nabla\times\mathbf{F}+\nabla\times\mathbf{G}$
7. $\nabla\cdot(f\mathbf{F}) = f(\nabla\cdot\mathbf{F})+\nabla f\cdot\mathbf{F}$
8. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F})-\mathbf{F}\cdot(\nabla\times\mathbf{G})$
9. $\nabla\cdot(\nabla\times\mathbf{F}) = 0$
10. $\nabla\times(f\mathbf{F}) = f(\nabla\times\mathbf{F})+\nabla f\times\mathbf{F}$
11. $\nabla\times(\nabla f) = 0$
12. $\nabla^2(fg) = f\left(\nabla^2 g\right)+2(\nabla f\cdot\nabla g)+g\left(\nabla^2 f\right)$
13. $\nabla\cdot(\nabla f\times\nabla g) = 0$
14. $\nabla\cdot(f\nabla g - g\nabla f) = f\left(\nabla^2 g\right)-g\left(\nabla^2 f\right)$
15. $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$

Since we shall use Identity 15 below, let me give a brief indication as to why it should be true. Both sides of this equation are vector fields (in the end); we shall look only at the x component

$$(\nabla \times (\nabla \times \mathbf{F}))_x = \left(\nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}\right)_x$$

Now the x-component of the right hand side is

$$\begin{aligned}
(RHS)_x &= \left(\nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}\right)_x \\
&= \nabla_x (\nabla \cdot \mathbf{F}) - \nabla^2 F_x \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_x \\
&= \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial F_z}{\partial x \partial z} - \frac{\partial^2 F_x}{\partial x^2} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} \\
&= \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial F_z}{\partial x \partial z} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2}
\end{aligned}$$

The x-component of the left hand side is

$$\begin{aligned}
(LHS)_x &= (\nabla \times (\nabla \times \mathbf{F}))_x \\
&= \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]_x \\
&= \frac{\partial}{\partial y}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \\
&= \frac{\partial^2 F_y}{\partial y \partial x} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_x}{\partial z \partial x} \\
&= \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial F_z}{\partial x \partial z} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} \\
&= (RHS)_x
\end{aligned}$$

So weve now confirmed the x-component of Identity 15.

2. Application: Maxwell's Equations

As an example of the utility of the identities listed in the preceding section, let us consider the equations governing the behavior of electric and magnetic fields. These are Maxwell's equation:

$$\begin{array}{ll}
\nabla \cdot \mathbf{E} = \frac{1}{4\pi\varepsilon_o}\rho(\mathbf{x}) & \text{(Gauss' Law)} \\
\nabla \cdot \mathbf{B} = 0 & \text{(Gauss' Law for Magnetic Field)} \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's Law)} \\
\nabla \times \mathbf{B} = \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} + \mu_o \mathbf{j}(\mathbf{x}) & \text{(Ampere's Law)}
\end{array}$$

Here $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ is the electric field strength at the point $\mathbf{x}$ at time t, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is the magnetic field strength at the point $\mathbf{x}$ at time t, $\rho(\mathbf{x})$ is the charge density at the point $\mathbf{x}$. ε_o is a constant called the *electric permitivity of the vacuum*, it is determined experimentally by measuring the force of attraction between electric charges

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_o \left\|\mathbf{r}\right\|^3}\mathbf{r}$$

and is equal to

$$\varepsilon_o = 8.85 \times 10^{-12} \text{ (colomb)}^2(\text{sec})^2(\text{kg})^{-1}(\text{meter})^{-3}$$

$\mathbf{j}(\mathbf{x})$ is the density of electrical current at the point $\mathbf{x}$, and μ_o is another experimentally determined constant. It is called the *magnetic permeability of the vacuum* and its value is

$$\mu_o = 1.26 \times 10^{-6} \text{ (coulomb)}^{-2}(\text{kg})(\text{meter})$$

In vacuum, where both the charge density and the current density are 0, we have

$$\nabla \cdot \mathbf{E} = 0 \tag{15.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{15.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{15.3}$$

$$\nabla \times \mathbf{B} = \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{15.4}$$

If we take the time derivative of the last equation we get

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

or

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right) = 0$$

Using Faraday's Law we have

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla \times (-\nabla \times \mathbf{E}) = 0$$

or

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\mu_o \varepsilon_o} \nabla \times (\nabla \times \mathbf{E}) = 0 \tag{15.5}$$

Let's now apply Identity 15 above:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\ &= 0 - \nabla^2 \mathbf{E} \end{aligned}$$

In the last step we have applied Gauss' Law in vacuum. Thus, Eqn. (15.5) can be written

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla^2 \mathbf{E} = 0$$

If we examine this equation component by component we find

$$\begin{aligned} \frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla^2 E_x &= 0 \\ \frac{\partial^2 E_y}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla^2 E_y &= 0 \\ \frac{\partial^2 E_z}{\partial t^2} - \frac{1}{\mu_o \varepsilon_o} \nabla^2 E_z &= 0 \end{aligned}$$

Each of these equations is of the form

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0.$$

This is the equation of a wave travelling in a 3-dimensional space with velocity c. Thus, each component of the electric field satisfies a wave equation with velocity

$$\begin{aligned} c &= \frac{1}{\sqrt{\varepsilon_o \mu_o}} \\ &= \frac{1}{\sqrt{(8.85 \times 10^{-12}\ (\text{colomb})^2(\text{sec})^2(\text{kg})^{-1}(\text{meter})^{-3})\,(1.26 \times 10^{-6}\ (\text{coulomb})^{-2}(\text{kg})(\text{meter}))}} \\ &= 2.99 \times 10^8 \text{m/s} \end{aligned}$$