LEX-PLUS-POWERS IDEALS

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Introduction

In the last several decades, researchers interested in Hilbert functions and free resolutions have been trying to understand the relationship between these two invariants. It is easy to give examples, for instance, of two ideals with the same Hilbert function but different graded Betti numbers. This raises the question:

Question 0.1. Given a Hilbert function for a cyclic module (i.e., a polynomial ring modulo a homogeneous ideal), what graded Betti numbers actually occur for modules with that Hilbert function?

Hilbert showed in his 1890 paper [Hilbert] that one can compute the Hilbert function from a graded free resolution (or simply the set of graded Betti numbers). Thus there is an (easy) combinatorial rule that any potential set of graded Betti numbers must satisfy. Even more promising is that given a Hilbert function, there is a sharp upper bound for the potential graded Betti numbers; that is, there exists a special ideal called a lex ideal whose quotient has uniquely largest graded Betti numbers among all cyclic modules attaining that Hilbert function. This fact, first proved independently by Bigatti and Hulett in characteristic zero and then later in characteristic p by Pardue, bounds the search for all sets of graded Betti numbers occurring for a given Hilbert function. Moreover, it leads one to consider the sets of graded Betti numbers for a fixed Hilbert function as a partially ordered set.

Unfortunately, not every potential element of this partially ordered set actually occurs, even if it satisfies the upper bound and combinatorial data. It is easy to construct Hilbert functions for which the poset has "gaps." Sometimes, simple counting arguments can explain this behavior. For example, lower bounds on the number of minimal generators an ideal must contain, or the number of d-th syzygies it must have by a certain degree are known, and this can account for some of the entries missing from the partial order. But these *ad hoc* arguments are not enough to determine completely which sets of graded Betti numbers occur for a particular Hilbert function. See, for example, [Evans-Richert] and Chapter 2 of [Francisco1] for further discussion.

One idea for getting more information about what graded Betti numbers occur for a given Hilbert function is to restrict to certain classes of ideals, thus restricting to a subposet of the partial order. In particular, one considers only cyclic modules with a fixed Hilbert function whose defining ideals contain regular sequences with elements lying in certain degrees, and wishes to show that the resulting subposet of graded Betti numbers has a unique largest element. The essence of the Lex-plus-powers (LPP) Conjecture is that a generalization of a lex ideal, called a lex-plus-powers ideal, gives the unique largest element of the smaller poset. The LPP Conjecture is a generalization of a conjecture of Eisenbud, Green, and Harris, who first identified the ideals in question as interesting in this situation. Proving the Lex-plus-powers Conjecture would constitute a major step forward in our understanding of the poset of resolutions. It has the additional benefit of recovering the Bigatti-Hulett-Pardue theorem in the Artinian characteristic zero case in a satisfying way.

Of course, simply knowing the existence of a unique largest element in a subposet will not always be useful in determining whether a potential set of graded Betti numbers actually occurs. Given the potential Betti diagram, one would need to identify in what degrees an ideal with that Betti diagram has to have a regular sequence. This can be done for some examples (see [Richert1]), but it would be difficult in general.

Our paper is organized as follows. First, we explain the partial order on resolutions for a given Hilbert function and introduce some terminology and notation we shall use. In Section 2, we discuss properties of lex ideals that LPP ideals are conjectured to generalize, surveying the classical work of Macaulay and the homological results of Bigatti-Hulett-Pardue. We introduce LPP ideals in Section 3 and prove some basic properties. In Section 4, we describe the Eisenbud-Green-Harris Conjecture and some cases that are known, and we do the same for its homological analogue, the LPP Conjecture, in Section 5. We conclude in Section 6 with some alternate statements of the EGH Conjecture and reductions of the LPP Conjecture.

We thank Irena Peeva for encouraging us to write this paper. We are also grateful to Susan Cooper, Graham Evans, Tony Geramita, Jeff Mermin, Juan Migliore, Uwe Nagel, Irena Peeva, Sindi Sabourin, and Mike Stillman for their work and helpful conversations over the years on the topics we survey here.

We dedicate this paper to Graham Evans, our advisor and mentor. We thank him for his time, patience, friendship, and advice through the years, and we wish him the best in his retirement. His influence will be evident throughout this work.

1. The partial order and some notation

In this section, we give an overview of the theory of the possible resolutions for a given Hilbert function. We introduce in detail the partial order we shall use throughout the paper and discuss some difficulties that arise in determining what resolutions occur for a particular Hilbert function.

Throughout, let $R = k[x_1, \ldots, x_n]$, where k is a field. (We shall use S to denote particular polynomial rings.) Given a graded module R/I, let $\beta_{i,j}^{R/I}$ be the (i, j) graded Betti number of R/I; that is, $\beta_{i,j}^{R/I}$ counts the number of minimal syzygies of degree j at step i of the resolution. Alternatively, note that

$$\beta_{i,j}^{R/I} = \dim_k \operatorname{Tor}_i^R(k, R/I)_j.$$

To investigate the question of what resolutions can occur for a given Hilbert function, we focus on the graded Betti numbers of modules, disregarding the maps.

Suppose I and J are homogeneous ideals in R with the same Hilbert function. We say that $\beta^{R/I} \leq \beta^{R/J}$ if and only if $\beta_{i,j}^{R/I} \leq \beta_{i,j}^{R/J}$ for all i and j. The inequality has to go the same way for each graded Betti number, so this is a strong condition. In particular, there are likely to be incomparable resolutions for a given Hilbert function (see Example 1.4).

Example 1.1. As a first example, consider ideals in the polynomial ring $S = k[a, b]$. Let $I = (a^2, b^2)$, and let $L = (a^2, ab, b^3)$ be ideals in S. Then I and L have the same

Hilbert function. However, their graded Betti numbers are different. The minimal graded free resolutions are:

$$0 \to S(-4) \to S(-2)^2 \to S \to S/I \to 0$$

$$0 \to S(-3) \oplus S(-4) \to S(-2)^2 \oplus S(-3) \to S \to S/L \to 0$$

To display the graded Betti numbers of modules, we shall often use the notation of Grayson and Stillman's computer algebra system Macaulay 2 [Grayson-Stillman]. The rows and columns are numbered starting from zero, with the columns representing successive steps in the resolution, and one can find $\beta_{i,j}$ in column i and row $j-i$ in the table. The graded Betti diagrams of S/I and S/L are:

S/I:	total:	1	2	1
	0:	1	.	.
	1:	.	2	.
	2:	.	.	1

S/L:	total:	1	3	2
	0:	1	.	.
	1:	.	2	1
	2:	.	1	1

Note that L has an extra generator and first syzygy of degree three not present in the minimal resolution of I. Both I and L have two generators of degree two and a first syzygy of degree four. Thus $\beta^{S/I} \leq \beta^{S/L}$.

Though there can be many sets of graded Betti numbers for a given Hilbert function, all will have the same alternating sum along the upward sloping diagonals in the graded Betti diagram. That is, if one holds the degree d constant and takes the alternating sum of the number of syzygies of degree d at each place in the minimal resolution, that number will be the same for any module with the same Hilbert function. This is a consequence of the following result that one can find in Stanley's "green book" [Stanley]. (Note that although we write the sum over d on the right-hand side as an infinite sum, the sum is really finite.)

Theorem 1.2. *If M is an $R = k[x_1, \dots, x_n]$-module, then*

$$\sum_{d=0}^{\infty} H(M,d)t^d = \frac{\sum_{d=0}^{\infty}\sum_{i=0}^{n}(-1)^i \beta_{i,d}^M t^d}{(1-t)^n}.$$

One can view Theorem 1.2 as describing how to read the Hilbert function off a resolution. Recall that the Hilbert function encodes vector space dimensions, and $\frac{1}{(1-t)^n}$ is the generating function for the Hilbert series of R. Then the vector space dimension of R/I in degree d is equal to the alternating sum of the dimensions of the degree d components of the free modules in a free resolution. Since the $\beta_{i,j}$ keep track of the degrees of the generators of these free modules, it follows that $H(R/I, d) = \frac{\sum_{i=0}^{n}(-1)^i \beta_{i,d} t^d}{(1-t)^n}$.

Example 1.3. Let $I = (a^2, b^2)$ and $L = (a^2, ab, b^3) \subset S = k[a,b]$ as in Example 1.1. Then the left-hand side is

$$\sum_{d=0}^{\infty} H(S/L,d)t^d = 1 + 2t + t^2.$$

The right-hand side is

$$\frac{1 + 0t - 2t^2 + (1-1)t^3 + t^4}{(1-t)^2} = 1 + 2t + t^2.$$

The computation for S/I is identical except that instead of having a $(1-1)t^3$ term in the numerator of the right-hand side, we just have $0t^3$, which is, of course, the same.

If one restricts to ideals in the polynomial ring in two variables, the partially ordered set behaves nicely. Charalambous and Evans showed that given a Hilbert function $\mathcal{H}$ for a module $k[a,b]/I$, there is a unique maximal element in the partial order, a unique minimal element, and all possibilities (that Theorem 1.2 allows) in between occur for some module [Charalambous-Evans1]. When one adds another variable, however, the situation need not be so simple.

Example 1.4. To illustrate the increasing complexity in more than two variables, we present an example of Charalambous and Evans [Charalambous-Evans1] that shows there may be incomparable minimal elements in the partially ordered set of resolutions for a given Hilbert function. Let $S = k[a,b,c]$, and consider the Hilbert function $H = (1,3,4,2,1)$. Let $I = (a^3, b^3, c^3, ac, bc)$, and let $J = (a^5, b^2, c^2, a^2b, a^2c)$. Then S/I and S/J both have Hilbert function H.

S/I:	total:	1	5	6	2
	0:	1	.	.	.
	1:	.	2	1	.
	2:	.	3	4	1
	3:	.	.	.	.
	4:	.	.	1	1

S/J:	total:	1	5	6	2
	0:	1	.	.	.
	1:	.	2	.	.
	2:	.	2	4	.
	3:	.	.	.	1
	4:	.	1	2	1

Clearly the resolutions of S/I and S/J are incomparable. If there is a resolution below both, the rank of the free module at step three in the resolution must be one. Since any module with Hilbert function H is Artinian, it is Cohen-Macaulay, and therefore a resolution below both S/I and S/J would be the resolution of a Gorenstein module. But H is not a symmetric Hilbert function, so we have a contradiction. Thus since there can be no resolution below both the resolutions of S/I and S/J, there are incomparable minimal elements in the partially ordered set.

Armed with this partial order, we can phrase a number of questions about what graded free resolutions occur for a given Hilbert function in terms of the structure of the partially ordered set. For example, we have seen that there is not always a unique minimal element in the poset. In the next section, we shall see that there is always a unique maximal element. Therefore the search for all possible sets of graded Betti numbers for a particular Hilbert function is a bounded problem. However, the behavior between the maximal element and the minimal element(s) can be unpredictable, and determining whether a particular potential set of graded Betti numbers can occur is generally difficult.

In this paper, we shall not consider all ideals with a given Hilbert function but rather only the ideals that contain a regular sequence in prescribed degrees. Recall that a sequence $f_1, \ldots, f_r$ of elements of $R = k[x_1, \ldots, x_n]$ is called a **regular sequence** if for all $i > 1$, f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$, and $(f_1, \ldots, f_r) \neq R$. The example we shall use most often is that if $a_i > 0$ for each i, then $x_1^{a_1}, \ldots, x_n^{a_n}$ is a regular sequence.

We will often be interested in the degrees of a regular sequence. Suppose $f_1, \ldots, f_n$ is a regular sequence, and $\deg f_i = a_i$ for all i. We say that $f_1, \ldots, f_n$

is an $(a_1, \ldots, a_n)$-sequence, which we will sometimes abbreviate as an $\mathbb{A}$-sequence. Additionally, if an ideal I contains a subideal $(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n$ is a regular sequence, we say that I is an $(a_1, \ldots, a_n)$-ideal. For example, in $k[a, b, c]$, $(a^3, b^3, c^3, a^2b, ab^2, bc^2)$ is a $(3, 3, 3)$-ideal because it contains the ideal (a^3, b^3, c^3). Note that it is also a $(3, 3, 4)$-ideal since (a^3, b^3, c^4) is a subideal.

2. Lex Ideals

In this section we discuss the fact that the partial order formed by the sets of graded Betti numbers of quotients attaining a given Hilbert function is sharply bounded. This requires identifying an ideal exhibiting the required maximal behavior. In order to do this, we need to give an order on the monomials of R.

Definition 2.1. The **lexicographic (lex) order** on R is the total order defined as follows. First we choose an order for the variables, $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots >_{\text{lex}} x_n$. Then we say that $x_1^{\alpha_1} \cdots x_n^{\alpha_n} >_{\text{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n}$ if $\alpha_i > \beta_i$ for the first index i such that $\alpha_i \neq \beta_i$. We will write $>$ to denote $>_{\text{lex}}$.

Example 2.2. If $S = k[x_1, x_2, x_3]$ then the monomials of degree 3 in descending lex order are $x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_1x_2x_3,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2,\ x_3^3$. We could also use the lex order to make comparisons across degrees, so $x_1x_2x_3 > x_2^4$, for instance, but we almost never do this because we are interested in homogeneous behavior.

We can now define a lex ideal.

Definition 2.3. A monomial ideal $I \subset R$ is called **lex** if for each $j \geq 0$, $(I \cap R_j)$ is generated as a k-vector space by the first $\dim_k(I \cap R_j)$ monomials of degree j in descending lex order.

Example 2.4. Suppose again that $S = k[x_1, x_2, x_3]$. Then

$$J = (x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_2^3)$$

is not lex because $(J \cap S_3)$ is not generated as a k-vector space by the first 5 monomials in lex order, $x_1^3 > x_1^2x_2 > x_1^2x_3 > x_1x_2^2 > x_1x_2x_3$. Also $I = (x_1^3, x_1^2x_2x_3)$ is not lex because $(I \cap S_4) = (x_1^4,\ x_1^3x_2,\ x_1^3x_3,\ x_1^2x_2x_3)$, is not generated as a k-vector space by the first $\dim_k(I \cap S_4) = 4$ monomials in lex order, $x_1^4 > x_1^3x_2 > x_1^3x_3 > x_1^2x_2^2$. The ideal $Q = (x_1^3,\ x_1^2x_2^2,\ x_1^2x_2x_3)$ repairs this, and since it is an easy exercise to show that an ideal is lex if and only if it is generated in degree j by the first $\dim_k(I \cap R_j)$ elements in lex order for degrees in which it has minimal generators, it follows that Q is lex.

As we shall discuss later in the section, given a Hilbert function $\mathcal{H}$, the lex ideal attaining $\mathcal{H}$ has everywhere largest graded Betti numbers (among all other ideals with Hilbert function $\mathcal{H}$). Of course, this takes for granted a crucial and as yet undiscussed step: that given a Hilbert function $\mathcal{H}$, there exists a lex ideal L such that $H(R/L) = \mathcal{H}$. This is a classical result of Macaulay.

Theorem 2.5 (Macaulay's theorem for Hilbert functions). *A function $\mathcal{H} : \mathbb{N} \to \mathbb{N}$ with $H(0) = 1$ is the Hilbert function of a cyclic module if and only if there is a lex ideal L such that $H(R/L) = \mathcal{H}$.*

It follows, of course, that given an ideal $I \subset R$, there is a lex ideal L such that $H(R/I) = H(R/L)$. In fact, once we know that such an ideal exists, it is very easily obtained (and is clearly unique). We simply take L to be the ideal generated by the first $H(I,d) = H(R,d) - H(R/I,d)$ d-forms in lex order for all d.

Example 2.6. Consider the ideal $I = (x_1x_3 + x_2^2,\ x_1x_2,\ x_3^3,\ x_2^2x_3,\ x_1^4)$. The Hilbert function of S/I is $H(S/I) = (1,3,4,2)$, while $H(S) = (1,3,6,10,15,\dots)$. Thus $H(I) = (0,0,2,8,15,\dots)$, so let $L = L_1 + L_2 + L_3 + L_4 + L_5 + \cdots$ where

$$\begin{aligned}
L_1 &= (0),\\
L_2 &= (x_1^2,\ x_1x_2),\\
L_3 &= (x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_1x_2x_3,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3),\\
L_4 &= (x_1^4,\ x_1^3x_2,\ x_1^3x_3,\ x_1^2x_2^2,\ x_1^2x_2x_3,\ x_1^2x_3^2,\ x_1x_2^3,\ x_1x_2^2x_3,\\
&\qquad x_1x_2x_3^2,\ x_1x_3^3,\ x_2^4,\ x_2^3x_3,\ x_2^2x_3^2,\ x_2x_3^3,\ x_3^4) = S_4\\
L_5 &= S_5\\
&\ \vdots
\end{aligned}$$

The ideal L is obviously lex as it is generated by lex segments, and we can easily check that the Hilbert function of S/L is $H(S/L) = (1,3,4,2)$ as expected. A minimal generating set for L is $L = (x_1^2,\ x_1x_2,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3,\ x_2x_3^3,\ x_3^4)$.

There are several equivalent formulations of Macaulay's theorem to which we now turn our attention. Each foreshadows various aspects of the conjectures towards which we are moving. We consider first the usual presentation of Macaulay's theorem, which is a statement about Hilbert function growth; the original Eisenbud-Green-Harris conjecture that we discuss in Section 4 was an attempt to generalize this behavior. Then, we will explore the version of Macaulay's theorem dealing with minimal generators. This motivates the idea that lex-plus-powers ideals should have largest graded Betti numbers, the assertion of the Lex-plus-powers Conjecture, considered in Section 5.

Proposition/Definition 2.7. *Let $a, d \in \mathbb{N}$ with $d > 0$. Then there are unique integers $a_d > a_{d-1} > \cdots > a_1 \geq 0$ such that $a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$. The sum $\binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$ is the* **Macaulay expansion** *of a with respect to d.*

Example 2.8. The Macaulay expansion for 17 with respect to 3, for instance, is

$$\binom{5}{3} + \binom{4}{2} + \binom{1}{1} = 10 + 6 + 1 = 17,$$

while the expansion for 16 with respect to 3 is

$$\binom{5}{3} + \binom{4}{2} + \binom{0}{1} = 10 + 6 + 0 = 16.$$

Given a and d, the algorithm for finding the a_i is quite simple. Let a_d be the largest integer such that $\binom{a_d}{d} \leq a$, and repeat. That is, let a_{d-1} be the largest integer such that $\binom{a_{d-1}}{d-1} \leq a - \binom{a_d}{d}$, a_{d-2} be the largest integer such that $\binom{a_{d-2}}{d-2} \leq a - \binom{a_d}{d} - \binom{a_{d-1}}{d-1}$, and so on.

These Macaulay expansions, whose existence can be proved using induction, turn out to be instrumental in describing exactly how much the Hilbert function of a cyclic module can grow from one degree to the next. This growth is controlled by the following arithmetic operation.

Definition 2.9. Let $a \in \mathbb{N}$ be given. Then for $d \in \mathbb{N}_+$, define $a^{\langle d \rangle}$ to be the integer

$$a^{\langle d \rangle} = \binom{a_d + 1}{d+1} + \binom{a_{d-1} + 1}{d} + \cdots + \binom{a_1 + 1}{2},$$

where

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$$

is the Macaulay expansion of a with respect to d.

Example 2.10. Consider again the Macaulay expansion from example 2.8. Then $17^{\langle 3 \rangle}$ is

$$\binom{5+1}{3+1} + \binom{4+1}{2+1} + \binom{1+1}{1+1} = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} = 15 + 10 + 1 = 26.$$

Macaulay's theorem can now be stated in the following terms.

Theorem 2.11 (Macaulay's theorem for Hilbert function growth)**.** *Let $\mathcal{H} : \mathbb{N} \to \mathbb{N}$ be a function with $\mathcal{H}(0) = 1$. Then $\mathcal{H}$ is the Hilbert function of some cyclic R-module if and only if $\mathcal{H}(d+1) \leq H(d)^{\langle d \rangle}$ for all $d \geq 1$.*

Example 2.12. Consider the sequence $\mathcal{H} = (1, 3, 6, 8, 9, 8, 9)$. Then we have

$$\begin{aligned}
\mathcal{H}(1) &= \binom{3}{1} \\
\mathcal{H}(2) &= \binom{4}{2} + \binom{0}{1} \\
\mathcal{H}(3) &= \binom{4}{3} + \binom{3}{2} + \binom{1}{1} \\
\mathcal{H}(4) &= \binom{5}{4} + \binom{4}{3} + \binom{1}{2} + \binom{0}{1} \\
\mathcal{H}(5) &= \binom{6}{5} + \binom{4}{4} + \binom{3}{3} + \binom{1}{2} + \binom{0}{1} \\
\mathcal{H}(6) &= \binom{7}{6} + \binom{5}{5} + \binom{4}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1}
\end{aligned}$$

Now note that $\mathcal{H}(0) = 1$ and

$$\begin{aligned}
\mathcal{H}(2) = 6 &\leq 6 = \binom{3+1}{1+1} = \mathcal{H}(1)^{\langle 1 \rangle} \\
\mathcal{H}(3) = 8 &\leq 10 = \binom{4+1}{2+1} + \binom{0+1}{1+1} = \mathcal{H}(2)^{\langle 2 \rangle} \\
\mathcal{H}(4) = 9 &\leq 10 = \binom{4+1}{3+1} + \binom{3+1}{2+1} + \binom{1+1}{1+1} = \mathcal{H}(3)^{\langle 3 \rangle} \\
\mathcal{H}(5) = 8 &\leq 11 = \binom{5+1}{4+1} + \binom{4+1}{3+1} + \binom{1+1}{2+1} + \binom{0+1}{1+1} = \mathcal{H}(4)^{\langle 4 \rangle} \\
\mathcal{H}(6) = 9 &\leq 9 = \binom{6+1}{5+1} + \binom{4+1}{4+1} + \binom{3+1}{3+1} + \binom{1+1}{2+1} + \binom{0+1}{1+1} = \mathcal{H}(5)^{\langle 5 \rangle}
\end{aligned}$$

We conclude that there is an ideal I which attains this Hilbert function. We can find such an ideal (once we know that Theorem 2.11 and Theorem 2.5 are equivalent), by finding following the algorithm given directly before (and utilized in) Example 2.6.

To go between Theorem 2.11 and Theorem 2.5 requires proving that if L is a lex ideal, then $H(R/L_{\leq d}, d+1) = H(R/L, d)^{\langle d \rangle}$, where $L_{\leq d}$ refers to the ideal generated by all forms in L of degree at most d. That lex ideals behave this way is not surprising given their very combinatorial description, but the equivalence gives no hint how to prove either version of the theorem. The best modern proof of Macaulay's theorem is by way of Theorem 2.11 and due to Green [Green]. He begins with the following result.

Theorem 2.13 (Green's theorem)**.** *Let I be an ideal of R (over an infinite field k) and $\binom{a_d}{d} + \cdots + \binom{a_1}{1}$ be the Macaulay expansion of $H(R/I, d)$. Then*

$$H\big(R/(I,h), d\big) \leq \binom{a_d - 1}{d} + \cdots + \binom{a_1 - 1}{1}$$

for a general linear form h.

The proof is by double induction on the dimension of the ring and the degree (see [Bruns-Herzog] for a beautiful algebraic version).

Example 2.14. Green's theorem allows one to bound below the drop in the Hilbert function of a module after one quotients by a (generic) element of degree one. For instance, consider the ideal

$$I = (a^2c,\ b^3,\ c^4,\ b^2c^3,\ abc^3,\ ab^2c^2,\ a^3b^2,\ a^4b,\ a^5).$$

The Hilbert function of S/I is $H(S/I) = (1, 3, 6, 8, 8)$, and we note that $H(S/I, 3) = 8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$. Green's theorem guarantees the existence of a linear form h such that $H(S/(I,h), 3) \leq 2 = \binom{4-1}{3} + \binom{3-1}{2} + \binom{1-1}{1}$. It is worth noting that none of $h = a$, $h = b$, or $h = c$ work in this instance—in each of these cases $H(S/(I+h), 3) = 3$. But $h = a + b$ behaves as required; that is, $H(S/(I, a+b), 3) = 2$.

Macaulay's theorem now follows as a corollary to Theorem 2.13. One observes from the short exact sequence

$$0 \to h(R/I)_d \to (R/I)_{d+1} \to (R/(I,h))_{d+1} \to 0,$$

where h is a general linear form, that $H(R/I, d+1) \leq H(R/I, d) + H(R/(I,h), d+1)$, applies Green's theorem to part of the expression on the right, and proceeds using purely numerical results about Macaulay expansions.

Macaulay's theorem is also equivalent to the following statement about generators.

Theorem 2.15. *Suppose that $I \subset R$ and L is a lex ideal such that $H(R/I) = H(R/L)$. Then L contains more minimal generators than I in each degree. In particular, $\beta^L_{1,j} \geq \beta^I_{1,j}$ for all $j \in \mathbb{N}$.*

Thus, extremal Hilbert function growth (as in the Hilbert function growth version of Macaulay's theorem) is equivalent to extremal behavior with respect to first graded Betti numbers. In 1994 Bigatti and Hulett proved independently (and very nearly simultaneously) the following extension of Macaulay's theorem in characteristic zero [Bigatti, Hulett]. Pardue showed the characteristic p case in 1998 [Pardue].

Theorem 2.16. *Let $I \subset R$ and L be the lex ideal such that $H(R/I) = H(R/L)$. Then $\beta^L_{i,j} \geq \beta^I_{i,j}$ for all $i = 1, \dots, n$ and $j \in \mathbb{N}$.*

Example 2.17. Consider again the ideal $I = (a^3, b^3, c^3, ac, bc)$ from Example 1.4. As previously mentioned, the Hilbert series of S/I is $H(S/I) = (1, 3, 4, 2, 1)$, and its Betti diagram is

S/I:	total:	1	5	6	2
	0:	1	.	.	.
	1:	.	2	1	.
	2:	.	3	4	1
	3:	.	.	.	.
	4:	.	.	1	1

The lex ideal with the same Hilbert series is $L = (a^2,\ ab,\ ac^2,\ b^3,\ b^2c,\ bc^3, c^5)$. As the Bigatti-Hulett-Pardue result guarantees, the Betti diagram of S/L,

S/I:	total:	1	7	10	4
	0:	1	.	.	.
	1:	.	2	1	.
	2:	.	3	5	2
	3:	.	1	2	1
	4:	.	1	2	1

is componentwise larger than that of S/I.

That Macaulay's theorem is a special case of the Bigatti-Hulett-Pardue theorem is clear from Theorem 2.15, the generator version of Macaulay's theorem.

The three proofs of this theorem, although quite different from each other, all resort at some point to a generic change of coordinates. This turns out, as we shall see in Sections 4 and 5, to be a major obstruction in generalizing the Bigatti-Hulett-Pardue theorem, and thus the technique is worth discussing here.

The first step in both Bigatti and Hulett's proofs is to specialize to the strongly stable case, where there is enough combinatorial data to solve the problem. The idea of the specialization is to pass from a given ideal I to a strongly stable ideal I' which has the same Hilbert function as I and graded Betti numbers which are no smaller. These three conditions are easily satisfied (in characteristic zero) by the so-called generic initial ideal of I. The generic initial ideal of I is the ideal obtained by first taking a generic change of basis, then forming the ideal consisting of initial forms (that is, consisting of the collection of monomials each of which is the largest element in the monomial order of some homogeneous element of I after the basis change). It is well known that the resulting ideal, called $Gin(I)$, is strongly stable, has the same Hilbert function as I, and has no smaller graded Betti numbers (because the graded Betti numbers are upper-semicontinuous [Galligo]).

Pardue avoids the characteristic zero requirement by taking a more complicated distraction. He first lifts a given ideal I to the ring $k[z_{i,j}]$, where $1 \leq i \leq n$ and $1 \leq j \leq J$ for J sufficiently large, obtaining

$$I^{(p)} = \left\{ \prod_{i=1}^{n} \prod_{j=1}^{\alpha_i} z_{i,j} \ : \ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ is a minimal generator of } I \right\}.$$

Then $I^{(p)}$ can be projected to R as follows: Choose a generic collection of linear forms $\{l_{i,j}\}$, where i, j vary with the indices of the $z_{i,j}$, let $\sigma_L(z_{i,j}) = l_{i,j}$, and compute $\sigma_L(I^{(p)}) \subset R$. Finally, make a generic change of basis on $\sigma_L(I^{(p)})$, and pass to the initial ideal. Pardue is able to show that the Hilbert function is maintained by this process, the graded Betti numbers are no smaller than before, and that after finitely many iterations, the result is a lex ideal. It is quite a remarkable proof.

Now that we know that the partial order of sets of graded Betti numbers of ideals attaining a given Hilbert function is bounded, we can turn our attention to the smaller elements in this partial order. Theorem 1.2 is a fundamental result that tells us what possibilities to consider below the lex ideal. Of course, this still leaves a large number of possible resolutions to consider. We can, without too much difficultly, discover which of these are attainable by monomial ideals–one simply asks the computer to list the monomial ideals with the given Hilbert function and then checks their resolutions (of course, for larger problems, the computer may actually require an exceedingly long time to finish the calculation). This is not enough, however, as it is known that monomial ideals do not give all possible resolutions. There are several ad hoc methods which have been brought to bear in various situations [Peeva, Evans-Richert, Richert1], but without too much trouble one finds examples which defeat these ideas.

What needs to be done next is evidently to discover a finer structure on the poset of possible resolutions for a given Hilbert function. One idea is to attempt to filter this poset by regular sequence degrees. That is, given a Hilbert function $\mathcal{H}$ and a increasing list of integers $a_1 \leq \cdots \leq a_n$, we consider sets of graded Betti numbers corresponding to ideals attaining $\mathcal{H}$ and containing a regular sequence whose forms have degrees $a_1, \ldots, a_n$. The hope is that such a subset of the larger poset also has a unique largest element. The candidate conjectured to attain the unique largest graded Betti number turns out to be a natural generalization of a lex ideal, and it is to these ideals that we turn out attention in the next section.

3. Lex-plus-powers ideals

As we discussed in the previous section, studying the partially ordered set of all resolutions for a given Hilbert function is difficult. To work with a simpler object, we restrict our study to ideals that contain a regular sequence in prescribed degrees. In this section, we shall investigate Artinian ideals in $R = k[x_1, \ldots, x_n]$, requiring that they contain a regular sequence of maximal length in degrees $a_1 \leq \cdots \leq a_n$.

We begin by defining the natural analogue of lex ideals in this setting.

Definition 3.1. Let $a_1 \leq \cdots \leq a_n$ be positive integers. We call L an $(a_1, \ldots, a_n)$ $-$lex-plus-powers (LPP) ideal if:
(1) L is minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}$ and monomials $m_1, \ldots, m_l$, and
(2) If r is a monomial, $\deg r = \deg m_i$, and $r >_{lex} m_i$, then $r \in L$.

Example 3.2. We check that $L = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3)$ is a $(2,3,3)-$LPP ideal. It contains appropriate powers of the variables, and we need only check the second condition for the other two generators. Since x_1^3, $x_1^2x_2$, and $x_1^2x_3$ are all in L, L is an LPP ideal.

In general, one builds an LPP ideal by first forming the regular sequence of maximal length. This is the "plus powers" portion of the generating set. Then, in order to get the desired Hilbert function, one adds more generators in descending lexicographic order in each degree, the "lex" part of the ideal.

We begin our analysis of LPP ideals by developing some properties of LPP ideals that correspond to useful characteristics of lexicographic ideals. We obtain tools to assist in our analysis of the relationship among conjectures on LPP ideals.

First, we find a basis in each degree for a quotient by an LPP ideal.

Proposition 3.3. *Let L be an $(a_1, \dots, a_n)-$LPP ideal in $R = k[x_1, \dots, x_n]$. Then the smallest $H(R/L, d)$ monomials in lexicographic order in degree d not divisible by any $x_i^{a_i}$ form a basis for $(R/L)_d$.*

Proof. Suppose that $x^e = x_1^{e_1} \cdots x_n^{e_n} \in L$ is not divisible by any $x_i^{a_i}$. Let $x^d = x_1^{d_1} \cdots x_n^{d_n}$ be a monomial not divisible by any $x_i^{a_i}$ such that $\deg x^e = \deg x^d$ and $x^d >_{\text{lex}} x^e$. We show that $x^d \in L$. If x^e is a minimal generator of L, then $x^d \in L$ by the definition of an LPP ideal. If not, then there exists a minimal generator s of L with $s = x_1^{e_1-g_1} \cdots x_n^{e_n-g_n}$, each $g_i \geq 0$, and some $g_i > 0$. Let m be the monomial of degree $\deg s$ equal to

$$m = x_1^{d_1} \cdots x_{i-1}^{d_{i-1}} x_i^{r_i},$$

where i is chosen such that

$$d_1 + \cdots + d_{i-1} < \deg s \leq d_1 + \cdots + d_i$$

and such that $\deg m = \deg s$.

We show that $m \geq_{lex} s$. Since $\deg m = \deg s$, the definition of an LPP ideal would then imply that $m \in L$. Since m divides x^d, this proves that $x^d \in L$.

First, suppose that $d_1 = e_1 - g_1, \dots, d_{i-1} = e_{i-1} - g_{i-1}$. Then $r_i \geq e_i - g_i$ since $\deg m = \deg s$, and $m \geq_{lex} s$. Otherwise, for some $j < i$, we have $d_1 = e_1 - g_1, \dots, d_{j-1} = e_{j-1} - g_{j-1}$, and $d_j \neq e_j - g_j$. If $d_j < e_j - g_j$, then $d_1 \leq e_1, \dots, d_{j-1} \leq e_{j-1}$, and $d_j < e_j - g_j \leq e_j$. But this contradicts the fact that $x^d >_{lex} x^e$. Therefore $d_j > e_j - g_j$, and $m \geq_{lex} s$. □

Corollary 3.4. *Suppose that L_1 and L_2 are two $(a_1, \dots, a_n)-$LPP ideals such that $H(R/L_1, d) = H(R/L_2, d)$, and suppose that all the minimal generators of L_1 and L_2 that are not pure powers have degree at most d. Then $(L_1)_{d+1} = (L_2)_{d+1}$.*

Proof. We have $(L_1)_d = (L_2)_d$, and the only new generators that occur for either ideal in degree $d+1$ are possibly some x_i^{d+1}. Therefore $(L_1)_{d+1} = (L_2)_{d+1}$. □

In other words, specifying the a_i and the value of the Hilbert function in degree d fully describes the monomials in an $(a_1, \dots, a_n)-$LPP ideal in degree d. Consequently, this information determines the Hilbert function in degree $d+1$, assuming any minimal generators in degree $d+1$ are pure powers.

It is easy to see that every Artinian lex ideal is an LPP ideal. Hence given an Artinian Hilbert function $\mathcal{H}$ (i.e., a Hilbert function that is eventually zero), because there is a quotient by a lex ideal attaining $\mathcal{H}$, there is a quotient by an LPP ideal attaining $\mathcal{H}$. However, even if there is an ideal corresponding to $\mathcal{H}$ containing a regular sequence in degrees $a_1, \dots, a_n$, the following easy example shows there need not be an $(a_1, \dots, a_n)$-LPP ideal corresponding to $\mathcal{H}$.

Example 3.5. Let $R = k[x_1, \dots, x_n]$, and let $\mathcal{H}$ be the Hilbert function (1). Then if $\mathfrak{m} = (x_1, \dots, x_n)$, $R/\mathfrak{m}$ has Hilbert function $\mathcal{H}$, and it contains the regular sequence $x_1^2, \dots, x_n^2$. However, there is obviously no $(2, \dots, 2)$-LPP ideal attaining the Hilbert function (1).

Given an Artinian Hilbert function $\mathcal{H}$, it is easy to find all the LPP ideals corresponding to $\mathcal{H}$ using a computer algebra system. (There is, for example, a Macaulay 2 package available from the first author that will do the computations.) It is not obvious *a priori*, however, for which sequences $a_1 \leq \cdots \leq a_n$ there will be an $(a_1, \ldots, a_n)$-LPP ideal. The following result, first conjectured by Evans, gives a partial answer; it will appear in a forthcoming paper by the second author.

Theorem 3.6. *Let $a_1 \leq \cdots \leq a_n$, $b_1 \leq \cdots \leq b_n$, and $c_1 \leq \cdots \leq c_n$ be positive integers. Suppose that $a_i \leq b_i \leq c_i$ for all i. If there exist $(a_1, \ldots, a_n)$-LPP and $(c_1, \ldots, c_n)$-LPP ideals corresponding to the Hilbert function H, then there exists a $(b_1, \ldots, b_n)$-LPP ideal corresponding to H.*

In other words, the set of LPP ideals for a given Hilbert function satisfies a convexity property. The theorem is not hard to prove for ideals in two variables and amounts to looking at the difference in the minimal generating sets of an (a_1, a_2)-LPP ideal and an $(a_1, a_2 + 1)-$LPP ideal with the same Hilbert function. In more variables, the situation is more complicated.

Example 3.7. Let $S = k[a, b, c]$, and consider the Hilbert function $H = (1, 3, 3, 1)$. $S/(a^2, b^2, c^2)$ has Hilbert function H, and (a^2, b^2, c^2) is a $(2, 2, 2)$-LPP ideal. The lex ideal corresponding to H is $(a^2, b^3, c^4, ab, ac, b^2c, bc^2)$, which is a $(2, 3, 4)$-LPP ideal. By Theorem 3.6, there exist LPP ideals corresponding to H with power sequences $(2, 2, 3)$, $(2, 2, 4)$, and $(2, 3, 3)$.

Since LPP ideals are the analogue of lex ideals when we consider only ideals with a maximal length regular sequence in prescribed degrees, it is natural to investigate generalizations of Macaulay's Theorem and the Bigatti-Hulett-Pardue Theorem. If there are corresponding results in the new setting, LPP ideals should play the role of lex ideals, meaning they should have largest Hilbert function growth and graded Betti numbers. We make these observations precise in the next two sections.

4. The EGH Conjecture and its ramifications

Lex-plus-powers ideals were first conjectured to exhibit extremal behavior in [Eisenbud-Green-Harris1]. Eisenbud, Green, and Harris made this assertion while exploring the following geometric statement, known as the Generalized Cayley-Bacharach Conjecture:

Conjecture 4.1. *Let $\Omega \subset \mathbb{P}^r$ be a complete intersection of quadrics. Any hypersurface of degree t that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^r - 2^{r-t}$ must contain Ω.*

The Generalized Cayley-Bacharach Conjecture claims that a form of degree t which vanishes at $2^r - 2^{r-t} + 1$ (or more) points of a complete intersection in $\mathbb{P}^r$ defined by forms of degree 2 must vanish on all the points in that intersection.

Conjecture 4.1 is not known in very many cases. Eisenbud, Green, and Harris give a proof for $r \leq 7$ [Eisenbud-Green-Harris2]. When Ω is a hypercube, Riehl and Evans [Riehl-Evans], have shown that the bounds hold. Finally, Davis, Geramita, and Orecchia [Davis-Geramita-Orecchia] have cast the conjecture in the language of Gorenstein rings.

More interesting for the current discussion was Eisenbud, Harris, and Green's observation that the following stronger conjecture implies the Generalized Cayley-Bacharach Conjecture.

Conjecture 4.2 (Eisenbud, Green, Harris Conjecture in degree 2)**.** *Suppose that $I \subset R$ is an ideal such that I_2 contains a regular sequence of maximal length. Then $H(R/I, d+1) \le \binom{a_d}{d+1} + \binom{a_{d-1}}{d} + \cdots + \binom{a_1}{2}$, where $\binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$ is the unique Macaulay expansion for $H(R/I, d)$.*

Example 4.3. Consider the Hilbert function $(1, 4, 10, 8, 2)$, and note that the Macaulay expansion for 8 with respect to 4 is $\binom{5}{4} + \binom{3}{3} + \binom{2}{2} + \binom{1}{1}$. Because $2 \not\le 1 = \binom{5}{5} + \binom{3}{4} + \binom{2}{3} + \binom{1}{2}$, the conjecture (which is actually known for dimension ≤ 5, as we will discuss below) implies that there does not exist an ideal containing a $(2, 2, 2, 2)$-regular sequence and attaining the Hilbert function $(1, 4, 10, 8, 2)$.

The Macaulayesque flavor of this conjecture is made even stronger after observing that the growth described in Conjecture 4.2 is exactly that of a $(2, \ldots, 2)$-lex-plus-powers ideal.

At the very end of their paper, Eisenbud, Green, and Harris mention that restricting to maximal regular sequences in degree two is artificial, and that a similar statement should hold for regular sequences in arbitrary degrees. In [Richert2, Francisco2], this conjecture is recorded in the following form:

Conjecture 4.4 (Eisenbud, Green, Harris)**.** *Let $I \subset R$ contain an $\mathbb{A}$-regular sequence, and suppose there exists an $\mathbb{A}$-lex-plus-powers ideal L such that $H(R/I, d) = H(R/L, d)$. Then*

$$H(R/L_{\langle d\rangle}, d+1) \ge H(R/I, d+1).$$

The notation $L_{\langle d\rangle}$, given an $\mathbb{A}$-lex-plus-powers ideal L, denotes the ideal generated by $L_{\le d} + (x_1^{a_1}, \ldots, x_n^{a_n})$. We refer to this statement as the EGH Conjecture.

Example 4.5. Consider the Hilbert function $\mathcal{H} = (1, 3, 6, 8, 8)$. If EGH is true, then no ideal I with Hilbert function $H(S/I) = \mathcal{H}$ can contain a $(3, 4, 5)$-regular sequence. This follows by simply forming a $(3, 4, 5)$-lex-plus-powers ideal L with $H(R/L, d) = 8$ ($L = (a^2b^2, a^2bc, a^2c^2, a^3, b^4, c^5)$ will suffice), and computing that $H(S/L_{\langle 3\rangle}, 4) = 7 < 8$.

A few remarks are in order. First, we note that, like Conjecture 4.2 above, this is a statement which bounds Hilbert function growth by lex-plus-powers growth. It is possible to state the theorem numerically (see [Richert-Sabourin, Cooper-Roberts]), but the result is not so simple or binomial. Second, the hypothesis that there exists an $(a_1, \ldots, a_n)$-lex plus powers ideal such that $H(R/L, d) = H(R/I, d)$ is present only to rule out degenerative cases (see Example 3.5). This is made clear by the following equivalent form of the EGH Conjecture.

Conjecture 4.6. *Let $\mathcal{H}$ be a Hilbert function and $(a_1, \ldots, a_n) = \mathbb{A}$ be a list of degrees such that there exists an ideal I with $H(R/I) = \mathcal{H}$, I contains an $\mathbb{A}$-regular sequence, and I does not contain a $(b_1, \ldots, b_n) = \mathbb{B}$-regular sequence with $b_i \le a_i$ for all $i = 1, \ldots, n$ and $b_j < a_j$ for some $j \in \{1, \ldots, n\}$. Then there is an $\mathbb{A}$-lex-plus-powers ideal attaining $\mathcal{H}$.*

Thus, according to the conjecture, if I contains an $\mathbb{A}$-regular sequence, then an $\mathbb{A}$-lex-plus-powers ideal with Hilbert function equal to $H(R/I)$ fails to exist only if there is some $\mathbb{B} < \mathbb{A}$ such that I contains an $\mathbb{B}$-regular sequence, and there does exist a $\mathbb{B}$-lex-plus-powers ideal with the same Hilbert function.

To round out the comparison to Macaulay's theorem, we note that the EGH is also equivalent to the following conjecture (and the proof of this equivalence is exactly the same as in the lex case).

Conjecture 4.7. *Suppose that I contains an $\mathbb{A}$-regular sequence and that there exists an $\mathbb{A}$-lex-plus-powers ideal L such that $H(R/I) = H(R/L)$. Then $\beta^L_{1,j} \geq \beta^I_{1,j}$ for all $j \in \mathbb{N}$.*

Example 4.8. Let I be an ideal in $S = k[a, b, c]$ with Hilbert function $H(S/I) = (1, 3, 5, 4, 1)$ and suppose that I contains a $(2, 3, 3)$-regular sequence. If EGH is true, then we claim that I does not have any minimal generators of degree 5. This is because $L = (a^2, ab^2, c^3, b^3, abc^2)$ is $(2, 3, 3)$-lex-plus-powers, has Hilbert function $(1, 3, 5, 4, 1)$, and contains no 5-forms.

One can show, when k has characteristic zero, that lex ideals contain the latest possible regular sequences. In particular, if we think of a lex ideal L as an $\mathbb{A}$-lex-plus-powers ideal, then any ideal with Hilbert function $H(R/L)$ contains an $\mathbb{A}$-regular sequence. (See [Richert2] for a proof.) Thus Macaulay's theorem in the characteristic zero Artinian case is simply a special case of EGH.

EGH is known in very few cases. One can give a combinatorial proof for monomial ideals with a theorem of Clements and Lindsröm [Clements-Lindström]. It is also known to hold for $n = 2$ variables [Richert2, Richert-Sabourin]. One notes that Gotzmann's persistence theorem [Gotzmann] forces $H(S/I, i) > H(S/I, i+1)$ for all $i \geq d$ if $I \subset S = k[a, b]$ is such that I_d contains a maximal regular sequence; then one demonstrates that the consecutive drop in the Hilbert function of the corresponding lex-plus-powers ideal L in degrees $i \geq d$ can be bounded above by one. The proof, unfortunately, does not extend to three variables. In addition, there are a few other, even smaller cases (for instance, the second author has demonstrated in unpublished work that the conjecture in degree 2 holds for $n \leq 5$).

Recently, Cooper has done some work in a more geometric direction. Consider a finite set of distinct points in $\mathbb{P}^2$ formed by a complete intersection of type (a, b), written $\mathrm{CI}(a, b)$. That is, the set of points is the zero set of a complete intersection ideal generated by two polynomials of degrees a and b. Given a Hilbert function $\mathcal{H}$, let $\Delta\mathcal{H}$ be the first difference function, so $\Delta\mathcal{H} = (1, \mathcal{H}(1) - \mathcal{H}(0), \mathcal{H}(2) - \mathcal{H}(1), \dots)$. Cooper showed in [Cooper] that $\Delta\mathcal{H}$ is the Hilbert function of a subset of a $\mathrm{CI}(a, b)$ if and only if $\Delta\mathcal{H}$ satisfies a growth condition given by the theorem Clements-Lindström, which amounts to the growth condition coming from the EGH Conjecture. Cooper also studies subsets of complete intersections $\mathrm{CI}(a, b, c)$ in $\mathbb{P}^3$ in her forthcoming Ph.D. thesis; she has similar results when $a = 2$, $a = 3$, $c \geq a + b + 1$, or $(a, b, c) = (4, m, n)$, where m and n are not both 4.

We explore a closely related conjecture for free resolutions in the next section.

5. The LPP Conjecture

The EGH Conjecture discussed in the last section is the analogue to Macaulay's Theorem. In this section, we introduce and explore the LPP Conjecture, which is the analogue of the Bigatti-Hulett-Pardue Theorem.

The Bigatti-Hulett-Pardue Theorem says that the lex ideal has the largest graded Betti numbers among all ideals with the same Hilbert function. The natural extension of that result is that an LPP ideal should have the largest graded Betti numbers among all ideals with the same Hilbert function and regular sequence in the same degrees. We give the precise statement below. Its origin is a bit murky, but it is perhaps best described as due to Charalambous and Evans and inspired by the work of Eisenbud, Green, and Harris. The conjecture first appeared in a paper of Evans and the second author [Evans-Richert].

Conjecture 5.1 (LPP Conjecture)**.** *Let $L \subset R$ be an $(a_1,\dots,a_n)-$LPP ideal. Suppose $I \subset R$ is a homogeneous ideal with the same Hilbert function that contains a regular sequence in degrees $a_1,\dots,a_n$. Then $\beta^{R/I} \leq \beta^{R/L}$.*

The LPP Conjecture is stronger than the EGH Conjecture, just as the Bigatti-Hulett-Pardue Theorem implies Macaulay's Theorem. See [Francisco1] for a detailed argument. Thus a solution to the LPP Conjecture would also prove something geometric, the Generalized Cayley-Bacharach Conjecture.

We give an example to illustrate the LPP Conjecture.

Example 5.2. Let $S = k[a,b,c]$, and let I be an ideal generated by generic polynomials of degrees two, three, three, and four. Let L be the $(2,3,3)-$LPP ideal (a^2, b^3, c^3, ab^2c). Then $H(S/I) = H(S/L) = (1,3,5,5,2)$. The LPP Conjecture asserts that $\beta^{S/I} \leq \beta^{S/L}$. The Betti diagrams of the two modules are the following:

S/I:	total:	1	4	5	2
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	.
	4:	.	.	1	2

S/J :	total:	1	4	6	3
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	1
	4:	.	.	2	2

The diagrams are the same except on the degree six diagonal, where the entries in the diagram of S/J are larger, so $\beta^{S/I} \leq \beta^{S/L}$.

There is substantial computational evidence for the LPP Conjecture, but proving it in its full generality is difficult. It is tempting to borrow from the proofs of Bigatti and Hulett in the lexicographic case; for example, instead of comparing an arbitrary ideal I to a lexicographic ideal, they consider the generic initial ideal of I. In characteristic zero, this gives a strongly stable ideal with graded Betti numbers the same or larger than those of I, and one has convenient formulas for the graded Betti numbers of strongly stable ideals. To use this approach with the LPP Conjecture, however, one needs some way to fix not only the Hilbert function but also the degrees of the regular sequence. These degrees can change when passing to the generic initial ideal, which makes this method difficult. Another possibility is to use work of Charalambous and Evans that describes how to compute the minimal resolution for LPP ideals [Charalambous-Evans2]. Unfortunately, this method can be hard to use to compare resolutions of LPP ideals to other ideals, partially because of some unpredictable ideal quotients that arise.

Our goal in this section is to describe some cases of the LPP Conjecture that are known. We shall sketch some of the methods used in the proofs of the special cases. Unfortunately, unlike the EGH Conjecture, the LPP Conjecture is not even known yet in the monomial case, so there is much room for further work. In the next section, we shall discuss some reductions and equivalences that may make it easier to attack the conjecture.

We begin with rings of low dimension. Consider the ring $S = k[x_1, x_2]$ first. Let L be an (a_1, a_2)-LPP ideal in S, and suppose I is an ideal with the same Hilbert function that contains a regular sequence in degrees a_1 and a_2. Because the EGH Conjecture is known for ideals in this ring, it follows that $\beta_{1,j}^{S/I} \leq \beta_{1,j}^{S/L}$.

By Theorem 1.2,

$$\beta_{1,j}^{S/I} - \beta_{2,j}^{S/I} = \beta_{1,j}^{S/L} - \beta_{2,j}^{S/L}.$$

Rearranging, we have

$$\beta_{2,j}^{S/L} - \beta_{2,j}^{S/I} = \beta_{1,j}^{S/L} - \beta_{1,j}^{S/I} \geq 0,$$

and thus $\beta^{S/I} \leq \beta^{S/L}$.

In three variables, the LPP Conjecture is known in the case that I is a monomial ideal. It follows from the fact that the EGH Conjecture is known for monomial ideals plus a reformulation of the EGH Conjecture in terms of socle dimensions. See Section 6, particularly Conjecture 6.1.

In summary, in low dimension, we have the following result.

Theorem 5.3. *Let L be an $(a_1, \ldots, a_n)$-LPP ideal, and suppose I is an ideal with the same Hilbert function that also contains a regular sequence in degrees $a_1, \ldots, a_n$. If L and I are ideals in $S = k[x_1, x_2]$, then $\beta^{S/I} \leq \beta^{S/L}$. If L and I are ideals in $T = k[x_1, x_2, x_3]$, and I is a monomial ideal, then $\beta^{T/I} \leq \beta^{T/L}$.*

Two other cases of the LPP Conjecture are known. First suppose that L is a complete intersection, which means that $L = (x_1^{a_1}, \ldots, x_n^{a_n})$. Let I be another $(a_1, \ldots, a_n)$-ideal with the same Hilbert function. Then I is minimally generated by elements $f_1, \ldots, f_n$ and $g_1, \ldots, g_r$, where $\deg f_i = a_i - d_i$, with $d_i \geq 0$ for each i, and the f_i form a regular sequence. But if any $d_i > 0$, then the Hilbert function of R/I will be too small, so $d_i = 0$ for each i. But this means that $I = (f_1, \ldots, f_n)$, a complete intersection with minimal generators in degrees a_i. Therefore $\beta^{R/I} = \beta^{R/L}$. Hence the LPP Conjecture is trivial for the case in which L is a complete intersection. We remark, however, that if one considers all ideals with a given Hilbert function (and not just those ideals with regular sequence in prescribed degrees), a complete intersection does not necessarily have uniquely minimal graded Betti numbers. The Hilbert function $(1, 3, 5, 6, 6, 5, 3, 1)$ has incomparable minimal elements in its partial order, including the resolution of $k[a, b, c]/(a^2, b^3, c^5)$. See Theorems 3.1 and 3.2 in [Richert1].

The next logical case to consider is that in which the LPP ideal L is an almost complete intersection. Then $L = (x_1^{a_1}, \ldots, x_n^{a_n}, m)$ is an $(a_1, \ldots, a_n)$-LPP ideal with $m \notin (x_1^{a_1}, \ldots, x_n^{a_n})$. The next result appears in [Francisco2].

Theorem 5.4. *Let L be an $(a_1, \ldots, a_n)$-LPP almost complete intersection in $R = k[x_1, \ldots, x_n]$, and let $I \subset R$ be another $(a_1, \ldots, a_n)$-ideal with the same Hilbert function. Then $\beta^{R/I} \leq \beta^{R/L}$.*

Example 5.5. Let $S = k[a, b, c]$, and let $L = (a^2, b^3, c^3, ab^2c)$, a $(2, 3, 3)$-LPP ideal. Let I be an ideal generated by generic polynomials of degrees two, three, three, and four. Then S/L and S/I both have Hilbert function $(1, 3, 5, 5, 2)$. Their graded Betti diagrams are:

S/L:	total:	1	4	6	3
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	1
	4:	.	.	2	2

S/I:	total:	1	4	5	2
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	.
	4:	.	.	1	2

The proof of Theorem 5.4 is somewhat involved, but the idea is relatively simple, and we sketch it here. See [Francisco2] for more details. We need a way to compare the minimal graded free resolutions of an $(a_1, \dots, a_n)$-LPP ideal L in $S = k[x_1, \dots, x_n]$ to an ideal $I \subset S$ with the same Hilbert function and regular sequence in degrees $a_1, \dots, a_n$. Using some counting arguments, one can reduce to two cases. The easier case is when I is a complete intersection ideal that we can take to be $(x_1^{a_1}, x_2^{a_2-1}, x_3^{a_3}, \dots, x_n^{a_n})$. The minimal graded free resolution of I is easy to compute, and it is not hard to see that its graded Betti numbers are bounded above by those of the corresponding LPP ideal.

The other case one needs to consider is when I is an almost complete intersection of a certain form. We suppose that L is an $(a_1, \dots, a_n)$-LPP almost complete intersection ideal $(x_1^{a_1}, \dots, x_n^{a_n}, m)$, where $\deg m = d$. The ideal I to which we shall compare L is minimally generated by polynomials $f_1, \dots, f_n, g$, where the f_i form a regular sequence, $\deg f_i = a_i$, $\deg g = d$, and $(f_1, \dots, f_n) : (g)$ is a complete intersection.

We want to show that $\beta^{R/I} \le \beta^{R/L}$. The main difficulty is that it is hard to compute the minimal graded free resolution of an arbitrary ideal I even with the restrictions we have placed on I. Our approach is to avoid this problem by computing nonminimal graded free resolutions of both R/I and R/L. The shifts in the nonminimal resolutions we compute are identical for both modules, so we need only show that there is less nonminimality in the resolution of the LPP ideal. We illustrate the idea in an example, using a monomial ideal for I. For the extension to nonmonomial ideals, see Section 3 of [Francisco2].

Example 5.6. Let $L = (a^2, b^3, c^3, ab^2c) \subset S = k[a, b, c]$, the $(2, 3, 3)$-LPP ideal from Example 5.5. Let $I = (a^2, b^3, c^3, b^2c^2)$. Then I contains a regular sequence in degrees two, two, and three, and I and L have the same Hilbert function.

The nonminimal resolutions we want to construct come from mapping cones. We start with L. Note that we have the canonical short exact sequence of graded modules

$$0 \to S/(a, b, c^2)(-4) \to S/(a^2, b^3, c^3) \to S/L \to 0.$$

The minimal graded free resolutions of $S/H := S/(a, b, c^2)$ and $S/F := S/(a^2, b^3, c^3)$ are easy to compute because they are complete intersections. The Comparison Theorem says that there are maps between the modules in the two resolutions that make the diagram commutative:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & S & \xrightarrow{\partial_3^F} & S^3 & \xrightarrow{\partial_2^F} & S^3 & \xrightarrow{\partial_1^F} & S & \longrightarrow & S/F & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\uparrow & & {\scriptstyle C_2}\uparrow & & {\scriptstyle C_1}\uparrow & & \uparrow{\scriptstyle ab^2c} & & \uparrow{\scriptstyle ab^2c} & & \\
0 & \longrightarrow & S & \xrightarrow[\partial_3^H]{} & S^3 & \xrightarrow[\partial_2^H]{} & S^3 & \xrightarrow[\partial_1^H]{} & S & \longrightarrow & S/H & \longrightarrow & 0
\end{array}$$

Here, the ∂_i^F and ∂_i^H are the Koszul maps. We have suppressed the gradings to save room.

We are interested in detecting the nonminimality in the mapping cone resolution of S/L coming from the above diagram. Since the resolutions of S/F and S/H are minimal, no nonminimality will arise from the ∂ maps. The only possible nonminimality will come from having nonzero constants in the vertical maps. The far left map certainly induces nonminimality because it is just the identity map. To determine whether there is other nonminimality, we need to find C_2 and C_1. It is not hard to see that C_2 should give the relationship between the generators of H and F, and then it is easy to compute the two maps. They are:

$$C_2 = \begin{pmatrix} c & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} b^2c & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab^2 \end{pmatrix}.$$

Since there are no nonzero constants in these maps, there is no further nonminimality in the mapping cone resolution of S/L. If one keeps track of the gradings, one obtains the minimal graded free resolution of S/L shown in Example 5.5.

We note that the matrices C_i have some special properties. C_2 is a diagonal matrix with its nonzero entries the powers of the variables that appear in the additional generator ab^2c of L. Moreover, C_1 is just the matrix of 2×2 minors of C_2, and ab^2c is the determinant of C_2. This suggests a general strategy for finding the mapping cone resolution of almost complete intersection monomial ideals like L and I. We should compute the penultimate vertical map C in the diagram, and fill in the other vertical maps with the appropriate exterior powers of C.

To compare the resolution of I to that of L, we go through the same process for I. We have the canonical short exact sequence

$$0 \to S/(a^2,b,c)(-4) \to S/(a^2,b^3,c^3) \to S/I \to 0,$$

and it is easy to find the minimal graded free resolutions of the complete intersection ideals $F = (a^2,b^3,c^3)$ and $G = (a^2,b,c)$. The commutative diagram for I is:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & S & \xrightarrow{\partial_3^F} & S^3 & \xrightarrow{\partial_2^F} & S^3 & \xrightarrow{\partial_1^F} & S & \longrightarrow & S/F & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\uparrow & & {\scriptstyle D_2}\uparrow & & {\scriptstyle D_1}\uparrow & & \uparrow{\scriptstyle b^2c^2} & & \uparrow{\scriptstyle b^2c^2} & & \\
0 & \longrightarrow & S & \xrightarrow[\partial_3^H]{} & S^3 & \xrightarrow[\partial_2^H]{} & S^3 & \xrightarrow[\partial_1^H]{} & S & \longrightarrow & S/G & \longrightarrow & 0
\end{array}$$

Again, we have the obvious nonminimality from the identity map on the far left. The maps D_2 and D_1 are:

$$D_2 = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} b^2c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & b^2 \end{pmatrix}.$$

This time, we have a map with a nonzero constant in it. The 1 in the lower-right corner of D_2 signifies additional nonminimality in the mapping cone resolution of S/I, and thus the ranks of the free modules in the second and third positions in the minimal resolution of S/I should be one lower than the ranks in the resolution of S/L. One can verify this by looking at the Betti diagrams from Macaulay 2:

S/L:	total:	1	4	6	3
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	1
	4:	.	.	2	2

S/I:	total:	1	4	5	2
	0:	1	.	.	.
	1:	.	1	.	.
	2:	.	2	.	.
	3:	.	1	4	.
	4:	.	.	1	2

In more variables, the analysis is a bit more complicated, but the general idea is the same. Start with an almost complete intersection ideal of the form described above. Form the canonical short exact sequences as we did in the example, and write down the resulting commutative diagram. The penultimate vertical map on the left determines all the vertical maps, so any nonminimality is determined by a single matrix. From there, it is not hard to show that any nonminimality in the mapping cone resolution of the LPP ideal occurs in the mapping cone resolution of the other ideal. For substantial work in another direction on minimal graded free resolutions of almost complete intersections, see the paper of Migliore and Miró-Roig [Migliore-Miró-Roig].

Unfortunately, it seems difficult to extend the approach used in the proof of Theorem 5.4. Having the extra structure of the complete intersections is vital, making the situation much more complicated if one goes to ideals in n variables with $n + 2$ generators. Detecting nonminimality is thus much more difficult for ideals with more generators.

We conclude the section with a few open questions related to the LPP Conjecture. Since the conjecture is wide open, this is, of course, just a small sample of the possibilities.

Question 5.7. Are there simple combinatorial formulas for the graded Betti numbers of LPP ideals as there are for lex ideals?

Question 5.8. Is there a way to generalize Theorem 5.4 to almost complete intersection ideals I that do not have the same Hilbert function as an LPP almost complete intersection?

Question 5.9. Some interesting special cases of the LPP Conjecture: What if the a_i are all equal? Can we say anything interesting when we require I to be a monomial ideal (or some other special type of ideal)? Suppose L_1 is an $(a_1, \dots, a_n)$-LPP ideal, and L_2 is an $(a_1 + b_1, \dots, a_n + b_n)$-LPP ideal with the same Hilbert function, where all $b_i \geq 0$. Is $\beta^{R/L_1} \leq \beta^{R/L_2}$?

Remark 5.10. After we wrote the original draft of this paper, Mermin, Peeva, and Stillman proved the LPP Conjecture for ideals containing the squares of the variables [Mermin-Peeva-Stillman]. Their approach is to reduce to the case in which I is a squarefree Borel ideal plus the squares of the variables. As the authors note, it is not *a priori* clear that this reduction is possible since taking the generic initial ideal does not fix the degrees of the maximal length regular sequence in I. Thus even if I has the squares of the variables among its minimal generators, $\text{gin}(I)$ may not. Even with this reduction, the proof is not at all easy and requires considerable care. Mermin, Peeva, and Stillman use compression, a method Macaulay developed, which has been a fruitful approach to questions about Hilbert functions. See [Mermin] for more information on compression and [Mermin-Peeva] for interesting approaches to studying the Hilbert functions (and graded Betti numbers) of ideals that avoid the usual computations with binomials.

6. Equivalences and reductions

There are several other equivalent statements of both EGH and LPP (apart from the Macaulayesque equivalences of EGH discussed in Section 4), as well as interesting reductions of EGH to which we now turn our attention.

The first and most obvious question is whether EGH and LPP might themselves be equivalent. It is not difficult to show that LPP implies EGH (especially in light of the generator version of EGH given as Conjecture 4.7). *A priori*, it seems that LPP must in fact be quite a bit stronger than EGH, but for $n \leq 3$ at least, we can show that this is not the case. For $n = 2$, that EGH implies LPP follows immediately from Theorem 1.2; see the discussion before Theorem 5.3.

The proof that EGH and LPP are equivalent for dimension $n = 3$ is interesting because it makes use of another as yet unmentioned form of EGH. In fact, an easy (and similar to the $n = 2$ case) application of Theorem 1.2 suffices if one notes that EGH is equivalent to the following statement.

Conjecture 6.1 (EGH for socles)**.** *Suppose that L is lex-plus-powers with respect to $\mathbb{A}$ for some $\mathbb{A} = (a_1, \ldots, a_n)$, and I is an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. Then $\beta^L_{n,j} \geq \beta^I_{n,j}$ for all j.*

We will refer to this conjecture as EGH for socles, because the essence of the statement is that lex-plus-powers ideals should have largest socles. The proof that EGH implies EGH for socles can be found in [Richert2] and relies on the fact that the socle elements in lex-plus-powers ideals are well behaved; that is, if one quotients by the socle elements of a given degree, one gets another lex-plus-powers ideal. Given an ideal with a purportedly larger socle, one quotients by socle elements and uses the fact that lex-plus-powers ideals have more generators (as we are assuming EGH) to force a contradiction. The proof that EGH for socles implies EGH (found in [Richert-Sabourin]) follows from a mapping cone argument after demonstrating that, if $\underline{\boldsymbol{x}}$ is the $\mathbb{A}$-regular sequence in an $\mathbb{A}$-lex-plus-powers ideal L, then $(\underline{\boldsymbol{x}} : L)$ is again lex-plus-powers.

In the language of Betti diagrams, proving that EGH implies LPP requires showing that if the first column of the Betti diagrams of lex-plus-powers ideals are biggest, then so are all the columns. The equivalence of EGH and EGH for socles says that if lex-plus-powers ideals always have largest first columns, then they also have largest last columns, and vice-versa.

We can also make similar statements about the rows of Betti diagrams. It turns out that the Lex-plus-powers Conjecture is equivalent to the following statement, which we refer to as the Lex-plus-powers Conjecture for last rows.

Conjecture 6.2. *Suppose that $\mathcal{H}$ is a Hilbert function, I is an ideal containing an $\mathbb{A}$-regular sequence and attaining $\mathcal{H}$, $\rho_{\mathcal{H}}$ is the regularity of $\mathcal{H}$, and L is the $\mathbb{A}$-lex-plus-powers ideal attaining $\mathcal{H}$. Then $\beta^{L_{\mathcal{H},\mathbb{A}}}_{i,\rho_{\mathcal{H}}+i} \geq \beta^I_{i,\rho_{\mathcal{H}}+i}$ for $i = 1, \ldots, n$.*

In terms of the Betti diagrams, this conjecture is easier to describe. If I contains an $\mathbb{A}$-regular sequence and L is $\mathbb{A}$-lex-plus-powers such that $H(R/I) = H(R/L)$, then the entries in the last row of the Betti diagram of L are conjectured to be componentwise larger than those in the last row of the Betti diagram of I. That this is equivalent to LPP is quite surprising, but the proof is not difficult. Simply note that adding the $(\rho_{\mathcal{H}} - 1)$-st power of the maximal ideal to each of I and L perturbs only the last two rows of their Betti diagrams, whence induction and the usual application of Theorem 1.2 finish the proof.

In fact, one can show (using the same argument), that the first t columns of the Betti diagrams of lex-plus-powers ideals are always larger if and only if the first t entries in their last rows are always larger. This proves, in particular, that EGH is equivalent to the following.

Conjecture 6.3. *Suppose that I contains an $\mathbb{A}$-regular sequence, L is $\mathbb{A}$-lex-plus-powers, $H(R/I) = H(R/L)$, and ρ is the regularity of $H(R/I)$. Then $\beta^L_{1,\rho} \geq \beta^I_{1,\rho}$.*

In essence, this says that EGH is equivalent to lex-plus-powers ideals always having the largest number of minimal generators in the last possible degree (the last degree for which generators might occur is the first degree for which the Hilbert function is zero). In light of EGH for socles, we may give another equivalent statement of EGH.

Conjecture 6.4. *Suppose that I contains an $\mathbb{A}$-regular sequence, L is $\mathbb{A}$-lex-plus-powers, $H(R/I) = H(R/L)$, and ρ is the regularity of $H(R/I)$. Then $\beta^L_{n,\rho+n-1} \geq \beta^I_{n,\rho+n-1}$.*

That is, EGH is equivalent to lex-plus-powers ideals always having the largest number of socle elements in the last degree for which the Hilbert function is zero minus one (of course, in the final degree, the number of socle elements is simply the value of the Hilbert function in that degree, and hence is always equal in any comparison we might be making).

There is other motivation for the Eisenbud-Green-Harris Conjecture aside from its beauty in relation to Macaulay's theorem. In fact, it turns out that being lex-plus-powers is a fairly strong condition. By this we mean that, if any counterexample exists, then we can force the existence of other (seemingly) very unlikely counterexamples using inferences not difficult to make from the nature of LPP ideals. Recall, for instance, that an ideal is said to be *level* if it only contains socle elements in one degree (so an ideal I with Hilbert function $H(R/I) = \mathcal{H}$ is level if $\beta^I_{n,j} = 0$ for $j < \rho_{\mathcal{H}} + n$ and $\beta^I_{n,\rho_{\mathcal{H}}+n} \neq 0$). It turns out that if EGH is true for lex-plus-powers ideals (that is, if the $\mathbb{A}$-lex-plus-powers ideal attaining Hilbert function $\mathcal{H}$ has more generators in each degree than any lex-plus-powers ideal containing an $\mathbb{A}$-regular sequence and attaining $\mathcal{H}$) but false in general, then there exist level counterexamples (at least, in characteristic zero).

Theorem 6.5. *Suppose that EGH is false in the characteristic zero case in some ring $R = k[x_1, \ldots, x_n]$ but holds for lex-plus-powers ideals. Then there is an ideal $I \subset R =$, containing an $\mathbb{A}$-regular sequence, and an $\mathbb{A}$-lex-plus-powers ideal L with $\mathcal{H} = H(R/I) = H(R/L)$, such that*

(1) *$\beta^L_{1,j} \geq \beta^I_{1,j}$ for $j \leq \rho_{\mathcal{H}}$ where $\rho_{\mathcal{H}}$ is the regularity of $H(R/I)$*
(2) *$\beta^L_{1,\rho_{\mathcal{H}}+1} < \beta^I_{1,\rho_{\mathcal{H}}+1}$*
(3) *I is level*

Sketch of proof. If EGH fails, we may suppose that I is a minimal length counterexample containing an $\mathbb{A}$-regular sequence. That is, I is an ideal containing an $\mathbb{A}$-regular sequence, L is $\mathbb{A}$-lex-plus-powers such that $H(R/I) = H(R/L)$, and $\beta^L_{1,j} < \beta^I_{1,j}$ for some j. Writing $\rho_{\mathcal{H}}$ to be the regularity of $\mathcal{H} = H(R/I)$, it must be that $\beta^L_{1,j} \geq \beta^I_{1,j}$ for $j \leq \rho_{\mathcal{H}}$ and $\beta^L_{1,\rho_{\mathcal{H}}+1} < \beta^I_{1,\rho_{\mathcal{H}}+1}$. If this were not the case then let t to be the smallest integer such that $\beta^I_{1,t} > \beta^L_{1,t}$, and consider $I' = I + (x_1, \ldots, x_n)^t$ and $L' = L + (x_1, \ldots, x_n)^t$ respectively. For a new sequence of degrees (which we call $\mathbb{A}'$), it is now the case that I' contains an $\mathbb{A}'$-regular sequence, L' is lex-plus-powers with respect to $\mathbb{A}'$, $H(R/I') = H(R/L')$, and $\beta^{I'}_{1,t} > \beta^{L'}_{1,t}$, but the length of I' is strictly less than that of I, a contradiction.

To show that I is level we first suppose that in some degree $d < \rho_{\mathcal{H}} + 1$ the dimension of the socles of both R/I and R/L are nonzero. Let $s_I \in R_d$ be a preimage of a socle elements from R/I and let s_L be the largest (in lex order) monomial preimage of a dimension d socle element of R/L. Let $I' = I + (s_I)$ and $L' = L + (s_d)$. Then it can be shown that L' is lex-plus-powers with respect to $\mathbb{A}' \leq \mathbb{A}$, and that I' contains an $\mathbb{A}'$ regular sequence (and this is where the characteristic zero hypothesis is used). Obviously $H(R/I') = H(R/L')$, but this gives a contradiction, because adding socle elements of degree $d < \rho_{\mathcal{H}}$ has no effect on the number of degree $\rho_{\mathcal{H}} + 1$ generators. That is, I' is a smaller length counterexample.

So it is enough to show that $\dim \operatorname{soc}(R/L)_d \geq \dim \operatorname{soc}(R/I)_d$ for all d (because then if R/I has socle element in any degree except the last, so does R/L, and we can quotient by them as in the previous paragraph, obtaining a smaller counterexample). Suppose not. Then for some $d \leq \rho_{\mathcal{H}}$, $\dim \operatorname{soc}(R/L)_d < \dim \operatorname{soc}(R/I)_d$. Let S_L be the socle of S/L in degree d, let S_I be $\dim(S_L)$ elements of the socle of R/I in degree d, let s_I be a degree d socle element of $S/(I + S_I)$ and s_L be the largest element of $S_d - (L + S_L)_d$. Finally, let $I' = I + S_I + (s_I) + (x_1, \ldots, x_n)^{d+1}$ and $L' = L + S_L + (s_L) + (x_1, \ldots, x_n)^{d+1}$. One can demonstrate that if L' contains an $\mathbb{A}'$-regular sequence, then I' does as well. Furthermore, it is apparent that $H(R/I') = H(R/L')$, and one can show (see [Richert2], Lemma 4.3 for the proof) that $L + S_L$ can have no minimal generators (except pure power generators) in degree $d+1$ (this because it has no socle elements in degree d). Note that $H(S/(I_S + I + (s_I)), d+1) > H(S/(L + S_L + (s_L)), d+1)$ because s_L is not a socle element. It can be shown (and this is where the hypothesis that EGH holds for lex-plus-powers ideals is used) that $H(S/(I_S + I + (s_I)), d+1) - H(S/(L + S_L + (s_L)), d+1)$ plus the number of regular minimal generators of $(I + S_I + (s_I))_{d+1}$ is strictly larger than the number of pure power minimal generators of $(L + D_L + (s_L))_{d+1}$ (this argument is somewhat involved). It follows that $\beta^{I'}_{1,d+1} > \beta^{L'}_{1,d+1}$, a contradiction as I' is then a counterexample of shorter length. □

Given work characterizing level algebras in codimension three (see for instance [Geramita-Harima-Migliore-Shin]) there is hope that progress might be made by considering potential counterexamples of this form.

We end with another striking special counterexample which must exist if EGH fails.

Theorem 6.6. *Suppose that EGH is false in some ring $R = k[x_1, \ldots, x_n]$. Then there is an ideal $I \subset R$ containing an $\mathbb{A}$-regular sequence $\{f_1, \ldots, f_n\}$ and an $\mathbb{A}$-lex-plus-powers ideal L with $\mathcal{H} = H(R/I) = H(R/L)$ such that*

(1) $\beta^I_{n,\rho_{\mathcal{H}}+n-1} > \beta^L_{n,\rho_{\mathcal{H}}+n-1}$ *(where $\rho_{\mathcal{H}}$ is the regularity of $H(R/I)$),*
(2) $I_{\leq \rho_{\mathcal{H}}-1} = (f_1, \ldots, f_n)_{\leq \rho_{\mathcal{H}}-1}$,
(3) $L_{\leq \rho_{\mathcal{H}}-1} = (x_1^{a_1}, \ldots, x_n^{a_n})_{\leq \rho_{\mathcal{H}}-1}$.

Sketch of proof. If EGH is false for some n, then there is an I containing an $\mathbb{A}$-regular sequence and an $\mathbb{A}$-lex-plus-powers ideal L such that $H(R/I) = H(R/L)$ and $\beta^L_{1,\rho_{\mathcal{H}}+1} < \beta^I_{1,\rho_{\mathcal{H}}+1}$. Let $\underline{\boldsymbol{y}}$ denote an $\mathbb{A}$-regular sequence in I and $\underline{\boldsymbol{x}}$ denote the minimal monomial regular sequence in L. Then $L' = (\underline{\boldsymbol{x}} : L)$ is $\mathbb{A}'$-lex plus powers for some $\mathbb{A}' \leq \mathbb{A}$ and $I' = (\underline{\boldsymbol{y}} : I)$ contains an $\mathbb{A}'$-regular sequence (see [Richert-Sabourin], section x for a proof of this). Let $I'' = I' + (x_1, \ldots, x_n)^{(\sum_{i=1}^n a'_i)-(\rho_{\mathcal{H}}+1)+1}$ and $L'' = L' + (x_1, \ldots, x_n)^{(\sum_{i=1}^n a'_i)-(\rho_{\mathcal{H}}+1)+1}$. If we write $\mathbb{A}''$ to be the degrees of the minimal monomial sequence in L'', then it is easy to see that L'' is lex-plus-powers with respect to $\mathbb{A}''$, I contains an $\mathbb{A}''$-regular sequence, and $H(R/L'') = H(R/I'')$. Furthermore, by considering closely the mapping cone, one can conclude that L'' and I'' satisfy conditions (1) - (3) as required. This completes the sketch. □

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