

EQUIDISTRIBUTION AND THE HEIGHTS OF TOTALLY REAL AND TOTALLY p -ADIC NUMBERS

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ABSTRACT. C.J. Smyth was among the first to study the spectrum of the Weil height in the field of all totally real numbers, establishing both lower and upper bounds for the limit infimum of the height of all totally real integers and determining isolated values of the height. Later, Bombieri and Zannier established similar results for totally p -adic numbers and, inspired by work of Ullmo and Zhang, termed this the Bogomolov property. In this paper, we use results on equidistribution of points of low height to generalize both Bogomolov-type results to a wide variety of heights arising in arithmetic dynamics.

1. INTRODUCTION AND RESULTS

Recall that an algebraic number is said to be *totally real* if all of its Galois conjugates lie in the field \mathbb{R} under any choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For example, if ζ is a root of unity, then $\alpha = \zeta + \zeta^{-1}$ is a totally real number. Schinzel [22] proved the following:¹

Theorem (Schinzel 1973). *Let $\alpha \in \mathbb{Q}^{\text{tr}}$, $\alpha \neq 0, \pm 1$ be a totally real number. Then*

$$h(\alpha) \geq h\left(\frac{1 + \sqrt{5}}{2}\right) = \frac{1}{2} \log \frac{1 + \sqrt{5}}{2} = 0.2406059\dots$$

where h denotes the absolute logarithmic Weil height.

A natural generalization of the concept of a totally real number is that of a *totally p -adic number*, that is, an algebraic number α all of whose Galois conjugates lie in \mathbb{Q}_p for any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, where \mathbb{C}_p denotes the completion of the algebraic closure of \mathbb{Q}_p . Equivalently, we can ask that the minimal polynomial of α split over \mathbb{Q}_p . Note that unlike \mathbb{C}/\mathbb{R} , the extension $\mathbb{C}_p/\mathbb{Q}_p$ is of infinite degree. Bombieri and Zannier [9] proved the following analogue of the results of Smyth and Flammang:

Theorem (Bombieri and Zannier 2001). *Let L/\mathbb{Q} be a normal extension (possibly of infinite degree) and S is the set of finite rational primes such that L_p/\mathbb{Q}_p is*

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¹In fact Schinzel's result originally implied this result for integers, however, it is easy to see that it generalizes to totally real nonintegers as well; cf. e.g., [17, 18].

Galois (in particular, of finite degree), then

$$\liminf_{\alpha \in L} h(\alpha) \geq \frac{1}{2} \sum_{p \in S} \frac{\log p}{e_p(p^{f_p} + 1)}$$

where e_p and f_p denote the ramification and inertial degrees of L_p/\mathbb{Q}_p , respectively.

Bombieri and Zannier, in analogy to work of Ullmo and Zhang on the Bogomolov conjecture (see [26, 27, 25]), termed this property that there is a lower bound away from zero for the height of all numbers not roots of unity in the field the *Bogomolov property*. Let us make the following more precise definitions. Let K be a number field and $\varphi \in K(z)$ a rational map of degree at least 2, and denote by h_φ its associated *canonical height* following [10]. We can then generalize the notion of Bombieri and Zannier and ask which extensions F/K satisfy an analogue of the Bogomolov property with respect to the height h_φ . Specifically:

Definition 1. We say that a field F/K satisfies the *strong Bogomolov property with respect to h_φ* if

$$\liminf_{\alpha \in \mathbb{P}^1(\overline{K})} h_\varphi(\alpha) > 0.$$

We say F/K satisfies the *weak Bogomolov property with respect to h_φ* if

$$\liminf_{\substack{\alpha \in \mathbb{P}^1(\overline{K}) \\ h_\varphi(\alpha) > 0}} h_\varphi(\alpha) > 0.$$

We note that there well-known examples of fields which satisfy the weak Bogomolov property, but not the strong, specifically, Amoroso and Dvornicich [1] showed that the maximal abelian extension \mathbb{Q}^{ab} satisfies the weak Bogomolov property, while clearly since it contains all of the roots of unity, it does not satisfy the strong Bogomolov property.

The strong Bogomolov property, unlike the weak, seems to be intimately connected to equidistribution of small points. This is seen as well in the Theorem of Zhang's on points on an abelian variety defined over the maximal totally real extension [27]; specifically, Zhang shows that if A is an abelian variety defined over a number field K , and \hat{h} be a Néron-Tate height associated to a symmetric ample line bundle on A , then the set of points $A(K\mathbb{Q}^{\text{tr}})$ satisfies the following strong Bogomolov property:

$$\liminf_{x \in A(K\mathbb{Q}^{\text{tr}})} \hat{h}(x) > 0.$$

We will establish an analogous result for the canonical heights associated to iteration of rational maps on \mathbb{P}^1 . Since we wish to allow arbitrary number fields as our base fields, we will make the following generalization of the notions of totally real and totally p -adic:

Definition 2. Fix a base number field K , and let S be a set of places of K . For each $v \in S$, we choose a Galois extension L_v/K_v . We say that α is *totally L_S/K* if, for each $v \in S$, all of the K -Galois conjugates of α lie in L_v for each embedding $\overline{K} \hookrightarrow \mathbb{C}_v$.

Notice that the set of all totally L_S/K numbers forms a field L which is a normal extension of K .

Our main result in this paper is a broad generalization of these results to the dynamical heights arising from the iteration of rational functions. We give a simple measure-theoretic criterion which characterizes the situations above.

Theorem 1. *Let φ be a rational map defined over a number field K , and fix a finite set S of places of K and corresponding Galois extensions L_v/K_v for each $v \in S$. Let L be the field of numbers all of whose K -Galois conjugates lie in L_v for each $v \in S$, $\mu_{\varphi,v}$ be the v -adic canonical measure associated to φ , and \overline{L}_v the topological closure of L_v in the Berkovich projective line $\mathbb{P}^1(\mathbb{C}_v)$.*

If there exists a place $v \in S$ for which

$$\mu_{\varphi,v}(\overline{L}_v) < 1,$$

then L satisfies the strong Bogomolov property with respect to h_φ , that is,

$$\liminf_{\alpha \in L} h_\varphi(\alpha) > 0.$$

Further, if φ is a polynomial, then the converse is also true, namely, if L satisfies the strong Bogomolov property with respect to h_φ , then there exists a place v of L such that $\mu_{\varphi,v}(\overline{L}_v) < 1$.

In specific case of the maximal totally real field of \mathbb{Q} , an independent proof which does not require the assumption that the φ is a polynomial in the converse is due to Pottmeyer [19].

Before we prove our theorem, let us state a few of the interesting corollaries of our result:

Corollary 2. *Let φ/K and L_S/K be as in the theorem, and suppose there exists a place $v \in S$ such that $\mu_{\varphi,v}(\overline{L}_v) < 1$. Then the set $\text{PrePer}_\varphi(L)$ of preperiodic points of φ which are defined over the field L of all totally L_S/K numbers is finite.*

Corollary 3. *Let φ/K and L_S/K be as in the theorem, and L the field of all totally L_S/K numbers. If there exists a sequence of distinct algebraic numbers $\{\alpha_n\} \subset L$ such that $h_\varphi(\alpha_n) \rightarrow 0$, then $\mu_{\varphi,v}(\overline{L}_v) = 1$ for every place $v \in S$.*

Corollary 4. *Let L_p/\mathbb{Q}_p be a Galois extension with $L_p = \mathbb{R}$ if $p = \infty$. Then*

$$\liminf_{\substack{\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}}) \\ \alpha \text{ is totally } L_p}} h(\alpha) > 0.$$

We note that for all but finitely many primes (i.e., those of good reduction), the local dynamics of φ are trivial and resemble those of the standard height, so in particular we get the following:

Corollary 5. *Let φ be a rational map defined over a number field K . Then for almost all finite places v of K , we have*

$$\liminf_{\substack{\alpha \in \mathbb{P}^1(\overline{K}) \\ \alpha \text{ is totally } L_v}} h_\varphi(\alpha) > 0.$$

for any Galois extension L_v/K_v .

Let E/K be an elliptic curve defined over a number field K given by

$$E : y^2 = x^3 + ax + b.$$

Baker and Petsche [2, Theorems 17, 21] proved that if L/\mathbb{Q} is a totally real field, or a field all of whose completions at the places lying over a prime p have a bounded ramification and inertial degrees, then the infimum and limit infimum of $\hat{h}(P)$ for points $P \in E(L)$ are bounded away from zero by certain explicit constants.² As an

²In fact, in the case of bounded ramification, Baker proved this earlier in [4, §5, Case 1].

easy application of our theorem, we can obtain the following qualitative version of these results:

Corollary 6. *Let K be a number field and E/K be an elliptic curve. Let S be a finite set of places of K which are either archimedean or for which E has good reduction. For each $v \in S$ we let L_v/K_v be a (finite) Galois extension, with the assumption that $L_v = K_v = \mathbb{R}$ if $v \mid \infty$. Then there exists a constant $c > 0$ depending only on E/K and the fields L_v such that*

$$\liminf_{\substack{P \in E(\overline{K}) \\ x(P) \text{ is totally } L_S/K}} \hat{h}(P) \geq c > 0.$$

We note that our hypothesis of good reduction is not always necessary, as the results of Baker and Petsche apply even in cases of semistable reduction. Naturally, this result as stated is strongest when we choose S to be a single place, however, as happens in the bounds obtained by Baker and Petsche (akin to what we see in the theorem of Bombieri and Zannier), the constant should grow as the number of places S increases. Determining the exact order of this growth would seem to be an interesting open question.

1.1. Background on arithmetic dynamics. Early work on the distribution of numbers by height focused on rational points, with results like the uniform distribution of the Farey fractions, culminating in quantitative generalization of the distribution of rational points at all places by Choi [12, 13]. Shortly afterwards in a slightly different direction, Bilu [8] formulated an equidistribution theorem for algebraic points of small height at the archimedean place, proving that the Galois conjugates of a sequence of numbers with Weil height tending to zero must equidistribute along the unit circle in \mathbb{C} in the sense of weak convergence of measures. This result was later generalized to abelian varieties by Szpiro, Ullmo, and Zhang [24] and used in the proof of the Bogomolov conjecture by Zhang [27]. Later, these results were vastly generalized to dynamical heights and nonarchimedean places using the techniques of potential theory on Berkovich analytic spaces independently by Baker and Rumely [6], Favre and Rivera-Letelier [15, 16], and Chambert-Loir [11]. The proofs of our main theorems are rooted in this equidistribution result.

We briefly recall now the dynamical equidistribution theorem. Let $\varphi \in K(z)$ be a rational function of degree $d \geq 2$ for K a number field. Then the *dynamical (or canonical) height associated to φ* , first introduced by Call and Silverman [10], is given by the Tate limit

$$h_\varphi(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{d^n} h(\varphi^n(\alpha))$$

where as usual $\varphi^n = \varphi \circ \dots \circ \varphi$ denotes the n -fold iteration. The dynamical height is characterized by the properties that

- (1) The quantity $|h_\varphi(\alpha) - h(\alpha)|$ is bounded by an absolute constant for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.
- (2) $h_\varphi(\varphi(\alpha)) = d \cdot h_\varphi(\alpha)$ for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.

The points $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ which satisfy $h_\varphi(\alpha) = 0$ are precisely the preperiodic points of φ , that is, α which satisfy $\varphi^n(\alpha) = \varphi^m(\alpha)$ for some $n > m \geq 0$. In the case of $\varphi(z) = z^2$, for which $h_\varphi = h$, the preperiodic points are precisely $0, \infty$, and the roots of unity. We refer the reader to [23] for more background on dynamical heights.

Recent work, particularly regarding the equidistribution theorem which we will discuss in more detail below, has made it clear that the proper setting to study nonarchimedean dynamics appears to be the *Berkovich analytic space* $\mathbb{P}^1(\mathbb{C}_v)$, where \mathbb{C}_v is the completion of an algebraic closure of K_v . For background on the Berkovich analytic space, we refer the reader to [7, 3]. While $\mathbb{P}^1(\mathbb{C}_v)$ is totally disconnected, not locally compact, and not spherically complete (there exist nested sequences of closed balls with empty intersection), the Berkovich space $\mathbb{P}^1(\mathbb{C}_v)$ is a compact, Hausdorff, and path-connected, and it contains $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subset. As a set, one can construct the affine Berkovich space $\mathbb{A}^1(\mathbb{C}_v)$ as the *Berkovich spectrum* consisting of all bounded multiplicative seminorms on the normed ring $\mathbb{C}_v[T]$, so each point in Berkovich space corresponds to such a seminorm. The set \mathbb{C}_v sits inside $\mathbb{A}^1(\mathbb{C}_v)$ as the evaluation seminorms $f \mapsto |f(x)|_v$ for each $x \in \mathbb{C}_v$. When $v \mid \infty$, this is the whole story: all bounded multiplicative seminorms come from evaluation maps at a point and $\mathbb{A}_{\text{Berk}}^1(\mathbb{C}) = \mathbb{C}$. For nonarchimedean v , however, there is a wealth of new points, most important amongst which in our study is the *Gauss point* $\zeta_{0,1}$, which corresponds to the multiplicative seminorm

$$f \mapsto \sup_{\substack{z \in \mathbb{C}_v \\ |z|_v \leq 1}} |f(z)|_v.$$

One can check by Gauss's lemma that this does in fact define a multiplicative seminorm on $\mathbb{C}_v[T]$. The Berkovich affine line is then endowed with the weakest topology for which the seminorms are continuous.

The equidistribution theorem states that any sequence of small points with respect to φ , that is, a sequence $\{\alpha_n\}$ of distinct numbers satisfying $h_\varphi(\alpha_n) \rightarrow 0$, is equidistributed with respect to a certain *canonical measure* $\mu_{\varphi,v}$ at each place v . More specifically, we recall the following result (see [6, 15]):

Theorem (Dynamical equidistribution theorem). *Let φ be a rational map of degree at least 2 defined over the number field K . Let $\{\alpha_n\}$ be a sequence of distinct algebraic numbers satisfying $h_\varphi(\alpha_n) \rightarrow 0$ and let $G_K = \text{Gal}(\overline{K}/K)$. Let v be a place of K , and for each α_n we define the probability measure on $\mathbb{P}^1(\mathbb{C}_v)$ supported equally on each of the G_K -conjugates of α_n :*

$$[\alpha_n] = \frac{1}{\#G_K\alpha_n} \sum_{z \in G_K\alpha_n} \delta_z$$

where δ_z denotes the Dirac measure at z . If $\mu_{\varphi,v}$ denotes the canonical measure on $\mathbb{P}^1(\mathbb{C}_v)$ associated to φ at v , then we have weak convergence of measures:

$$[\alpha_n] \xrightarrow{w} \mu_{\varphi,v} \quad \text{as } n \rightarrow \infty,$$

that is to say, for each continuous real or complex valued function f on $\mathbb{P}^1(\mathbb{C}_v)$, we have

$$\frac{1}{\#G_K\alpha_n} \sum_{z \in G_K\alpha_n} f(z) \rightarrow \int_{\mathbb{P}^1(\mathbb{C}_v)} f(z) d\mu_{\varphi,v}(z) \quad \text{as } n \rightarrow \infty.$$

At a finite place v of good reduction of φ , the measure $\mu_{\varphi,v} = \delta_{\zeta_{0,1}}$ is the Dirac measure supported at the Gauss point $\zeta_{0,1}$. We recall that any given rational φ has good reduction at all but finitely many places of K . Bilu's theorem [8] is the special case of this theorem for the squaring map $\varphi(z) = z^2$, for which $h_\varphi = h$ and $\mu_{\varphi,\infty}$ is the normalized Haar measure of the unit circle in \mathbb{C} . We note that already Bilu's theorem gives a rather vivid picture (Figure ??) of why totally real numbers

cannot have height tending to 0, as they clearly cannot equidistribute around the unit circle in \mathbb{C} .

2. PROOFS

Proof of Theorem 1. Suppose for the sake of contradiction that there existed a sequence $\{\alpha_n\} \subset \mathbb{P}^1(\overline{K})$ of distinct totally L_S/K algebraic numbers such that $h_\varphi(\alpha_n) \rightarrow 0$. Let $\mu_{\varphi,v}$ be the canonical measure and $G_K = \text{Gal}(\overline{K}/K)$. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v \subset \mathbb{P}^1(\mathbb{C}_v)$. Define the usual measures

$$[\alpha_n] = \frac{1}{\#G_K\alpha_n} \sum_{z \in G_K\alpha_n} \delta_z$$

where, for $z \in \mathbb{C}_v$, δ_z denotes the Dirac measure at z . Then the equidistribution theorem tells us that we have weak convergence of measures

$$[\alpha_n] \xrightarrow{w} \mu_{\varphi,v}.$$

Recall that we have natural continuous embeddings $\mathbb{P}^1(\mathbb{C}_v) \hookrightarrow \mathbb{P}^1(\mathbb{C}_v)$, and hence we can identify L_v as a subset of $\mathbb{P}^1(\mathbb{C}_v)$. By assumption, $\mu_{\varphi,v}(\overline{L_v}^c) = 1 - \mu_{\varphi,v}(\overline{L_v}) > 0$, where $\overline{L_v}^c$ denotes the complement of $\overline{L_v}$ in $\mathbb{P}^1(\mathbb{C}_v)$. Since $\mu_{\varphi,v}$ is a regular measure, there exists a compact set $E \subset \overline{L_v}^c$ such that $\mu_{\varphi,v}(E) > 0$ as well. As E and $\overline{L_v}$ are disjoint closed subsets of $\mathbb{P}^1(\mathbb{C}_v)$, which is a compact Hausdorff space, we know by Urysohn's lemma that there exists a continuous function $f : \mathbb{P}^1(\mathbb{C}_v) \rightarrow [0, 1]$ which takes the value 1 on E and 0 on $\overline{L_v}$. But then

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} f d[\alpha_n] = \frac{1}{\#G_K\alpha_n} \sum_{z \in G_K\alpha_n} f(z) = 0 \quad \text{for all } n,$$

while

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} f d\mu_{\varphi,v} \geq \mu_{\varphi,v}(E) > 0.$$

Thus, for this function f ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{P}^1(\mathbb{C}_v)} f d[\alpha_n] \neq \int_{\mathbb{P}^1(\mathbb{C}_v)} f d\mu_{\varphi,v}$$

but this is a contradiction to the weak convergence of measures, and completes the proof of the first part of the theorem.

Let us now prove the converse under the assumption that φ is a polynomial. We will prove that if at all places $v \in S$, $\overline{L_v} \subset \mathbb{P}^1(\mathbb{C}_v)$ has full $\mu_{\varphi,v}$ -measure, then there exists a sequence $\alpha_n \in \mathbb{P}^1(\overline{K})$ such that $h_\varphi(\alpha_n) \rightarrow 0$.

Since φ is a polynomial, it is a theorem (see for example [5, 14, 6]) that the v -adic Berkovich Julia sets $J_v(\varphi)$ are compact in $\mathbb{A}^1(\mathbb{C}_v)$ and that the v -adic logarithmic capacity of each is given by

$$\gamma_{\infty,v}(J_v(\varphi)) = |a_d|_v^{-1/(d-1)}$$

where we have denoted by a_d the leading coefficient of φ , that is, $\varphi(z) = a_d z^d + O(z^{d-1})$. Let

$$\mathbb{J} = \prod_v J_v(\varphi)$$

be the adelic Berkovich Julia set. Then the (normalized) adelic logarithmic capacity $\gamma_\infty(\mathbb{J}) = \prod_v \gamma_{\infty,v}(J_v) = 1$ by the product formula. Our basic strategy will be to use the Fekete-Szegő theorem with splitting conditions due to Rumely ([21, Theorem

2.1] and [3, §6.7, 7.8]; see also [20]) to generate a sequence $\alpha_n \in L$ with $h_\varphi(\alpha_n) \rightarrow 0$. Fix $\epsilon > 0$ and let S' be the union of S with the infinite places of K and the places of bad reduction of φ . Let $h_{\varphi,v}$ denote the usual v -adic local height associated to iteration of φ and consider the adelic set

$$\mathbb{U} = \prod_{v \notin S'} \mathcal{D}(0, 1) \times \prod_{v \in S'} h_{\varphi,v}^{-1}([0, \epsilon]).$$

where $\mathcal{D}(0, 1)$ is the closed v -adic Berkovich unit disc. Notice that for $v \notin S'$, $J_v(\varphi) = \{\zeta_{0,1}\} \subset \mathcal{D}(0, 1)$ is the usual Gauss point, and since $h_{\varphi,v}$ is nonnegative for each v , it follows that \mathbb{U} is an open adelic neighborhood of \mathbb{J} .

It now follows from [21, Theorem 2.1], since at least one (in fact all) archimedean places v we have $J_v(\varphi)$ is compact, that there are infinitely many totally L_S algebraic numbers all whose Galois conjugates lie in the open adelic neighborhood \mathbb{U} . Since any such number has height $h_\varphi(\alpha) \leq |S'| \cdot \epsilon$, by taking $\epsilon \rightarrow 0$ we can generate an infinite sequence of mutually distinct totally L_v algebraic numbers for all $v \in S$ with $h_\varphi(\alpha) \rightarrow 0$, which is the desired conclusion. \square

Proof Corollary 2. It is clear that the set of all totally L_S/K numbers form a field L which is normal over K . To see that the set of preperiodic points is finite, merely note that if an infinite number of such points existed, the limit infimum of the φ -canonical height would necessarily be 0, contradicting Theorem 1. \square

Proof of Corollary 3. This is immediate as it is the contrapositive of the first part of the theorem. \square

Proof of Corollary 4. Let $\varphi(z) = z^2$ and $L_\infty = \mathbb{R}$. Then the equilibrium measure of the (Berkovich) complex Julia set is $\mu_{\varphi,\infty} = \lambda$ is the normalized Haar measure of the unit circle S^1 in $\mathbb{C}^\times \subset \mathbb{P}_{\text{Berk}}^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$. Clearly, the topological closure of \mathbb{R} in $\mathbb{P}^1(\mathbb{C})$ is $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, but $\lambda(\overline{\mathbb{R}}) = \lambda(\overline{\mathbb{R}} \cap S^1) = \lambda(\{\pm 1\}) = 0$. Thus the theorem applies and we see that all totally \mathbb{R} numbers $\alpha \in \overline{\mathbb{Q}}$, that is to say, all totally real numbers which are not zero or roots of unity (the preperiodic points of φ) must have Weil height bounded away from zero by an absolute constant.

Now consider the case where $\varphi(z) = z^2$ and L_p/\mathbb{Q}_p is a Galois extension for a finite prime p . The equilibrium measure $\mu_{\varphi,p} = \delta_{\zeta_{0,1}}$ is the Dirac measure supported on the Gauss point $\zeta_{0,1}$. Now observe that $L_p \hookrightarrow \mathbb{P}^1(L_p) \hookrightarrow \mathbb{P}^1(\mathbb{C}_v)$. But $\mathbb{P}^1(L_p)$ is compact and the inclusion map is continuous, so $\mathbb{P}^1(L_p)$ is compact (and in particular, closed) in $\mathbb{P}^1(\mathbb{C}_v)$. Therefore the Berkovich closure $\overline{L_p} = L_p \cup \{\infty\}$, where ∞ is the usual point $\infty \in \mathbb{P}^1(\mathbb{C}_v)$, but this clearly implies that $\zeta_{0,1} \notin \overline{L_p}$. Therefore, $\mu_{\varphi,p}(\overline{L_p}) = 0 < 1$ and the theorem again applies. \square

Proof of Corollary 5. We merely note that for almost all places of K , φ has good reduction, which means that the Berkovich v -adic Julia set is the Gauss point. But then, as we argued in the proof of the above corollary, $\mu_{\varphi,v}(\overline{L_p}) = 0$, and so the theorem applies. \square

Before we prove Corollary 6, let us recall some background on elliptic curves and Lattès maps. As before, let E/K be an elliptic curve defined over a number field K given by

$$E : y^2 = x^3 + ax + b.$$

Recall that the *Lattès map* associated to the doubling map $[2] : E \rightarrow E$, $[2](P) = 2P$, is given by

$$\varphi(x) = \frac{x^4 - 2ax^2 - 8bx + a^2}{4x^3 + 4ax + 4b}.$$

It is characterized by the fact that the following diagram commutes:

$$\begin{array}{ccc} E(\overline{K}) & \xrightarrow{[2]} & E(\overline{K}) \\ x \downarrow & & \downarrow x \\ \mathbb{P}^1(\overline{K}) & \xrightarrow{\varphi} & \mathbb{P}^1(\overline{K}) \end{array}$$

where $x : E(\overline{K}) \rightarrow \mathbb{P}^1(\overline{K})$ is the map onto the x -coordinate. If $\hat{h} : E(\overline{K}) \rightarrow \mathbb{R}$ denotes the Néron-Tate height on E and h_φ is the dynamical height associated to the Lattès map, then

$$2\hat{h}(P) = h_\varphi(x(P)) \quad \text{for all } P \in E(\overline{K}).$$

Proof of Corollary 6. Let us first consider the proof for $p = \infty$ and $L_p = \mathbb{R}$. Recall that $E_{\text{tors}}(\mathbb{C})$ is dense in $E(\mathbb{C})$, so $x(E_{\text{tors}}(\mathbb{C}))$ is dense in $\mathbb{P}^1(\mathbb{C})$. It follows from this that the complex Julia set of φ is $J_\infty = \mathbb{P}^1(\mathbb{C})$. Thus, $\mu_{\varphi, \infty}(\mathbb{P}^1(\mathbb{R})) < 1$, and our theorem implies that $P \in E(L)$ satisfy

$$2\hat{h}(P) = h_\varphi(x(P)) \geq c$$

for some constant $c = c(L_p, E/K) > 0$.

Now suppose that $p \nmid \infty$. Let $\Delta = -16(4a^3 + 27b^2)$ be the discriminant of the elliptic curve. Recall that E has good reduction at v if and only if $|\Delta|_v = 1$. We simply observe that Δ and the resultant of φ , $\text{Res}(\varphi) = 256(4a^3 + 27b^2)^2$, satisfy $|\Delta|_v \neq 1 \iff |\text{Res}(\varphi)|_v \neq 1$, and thus E has good reduction if and only if φ does. But then for some $v \mid p$, we know that the equilibrium measure $\mu_{\phi, v} = \delta_{\zeta_{0,1}}$ is the usual Dirac measure at the Gauss point and $\mu_{\phi, v}(L_p) = 0$, so our theorem applies as before. \square

REFERENCES

- [1] F. Amoroso and R. Dvornicich. A lower bound for the height in abelian extensions. *J. Number Theory*, 80(2):260–272, 2000.
- [2] M. Baker and C. Petsche. Global discrepancy and small points on elliptic curves. *Int. Math. Res. Not.*, (61):3791–3834, 2005.
- [3] M. Baker and R. Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.
- [4] M. H. Baker. Lower bounds for the canonical height on elliptic curves over abelian extensions. *Int. Math. Res. Not.*, (29):1571–1589, 2003.
- [5] M. H. Baker and L.-C. Hsia. Canonical heights, transfinite diameters, and polynomial dynamics. *J. Reine Angew. Math.*, 585:61–92, 2005.
- [6] M. H. Baker and R. Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.
- [7] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [8] Y. Bilu. Limit distribution of small points on algebraic tori. *Duke Math. J.*, 89(3):465–476, 1997.

- [9] E. Bombieri and U. Zannier. A note on heights in certain infinite extensions of \mathbb{Q} . *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 12:5–14, 2001.
- [10] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.
- [11] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.
- [12] K.-K. Choi. Diophantine approximation on projective spaces over number fields. *Ph.D. dissertation, University of Texas at Austin*, 1996.
- [13] K.-K. Choi. On the distribution of points in projective space of bounded height. *Trans. Amer. Math. Soc.*, 352(3):1071–1111, 2000.
- [14] L. DeMarco and R. Rumely. Transfinite diameter and the resultant. *J. Reine Angew. Math.*, 611:145–161, 2007.
- [15] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.*, 335(2):311–361, 2006.
- [16] C. Favre and J. Rivera-Letelier. Corrigendum to: “Quantitative uniform distribution of points of small height on the projective line” (French) [Math. Ann. **335** (2006), no. 2, 311–361; mr2221116]. *Math. Ann.*, 339(4):799–801, 2007.
- [17] J. Garza. On the height of algebraic numbers with real conjugates. *Acta Arith.*, 128(4):385–389, 2007.
- [18] G. Höhn. On a theorem of Garza regarding algebraic numbers with real conjugates. *Int. J. Number Theory*, 7(4):943–945, 2011.
- [19] L. Pottmeyer. Fields with the bogomolov property. *Ph.D. dissertation, Univ. Regensburg*, 2012.
- [20] R. Rumely. The Fekete-Szegő theorem with local rationality conditions on curves (preprint).
- [21] R. Rumely. The Fekete-Szegő theorem with splitting conditions. II. *Acta Arith.*, 103(4):347–410, 2002.
- [22] A. Schinzel. On the product of the conjugates outside the unit circle of an algebraic number. *Acta Arith.*, 24:385–399, 1973. Collection of articles dedicated to Carl Ludwig Siegel on the occasion of his seventy-fifth birthday. IV.
- [23] J. H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.
- [24] L. Szpiro, E. Ullmo, and S. Zhang. Équirépartition des petits points. *Invent. Math.*, 127(2):337–347, 1997.
- [25] E. Ullmo. Positivité et discrétion des points algébriques des courbes. *Ann. of Math. (2)*, 147(1):167–179, 1998.
- [26] S.-W. Zhang. Positive line bundles on arithmetic surfaces. *Ann. of Math. (2)*, 136(3):569–587, 1992.
- [27] S.-W. Zhang. Equidistribution of small points on abelian varieties. *Ann. of Math. (2)*, 147(1):159–165, 1998.

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