Stochastic Dynamics and Equidistribution

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Algebraic Dynamics and its Connections to Difference and Differential Equations
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Dynamics of the iteration of a rational map

Suppose that $\varphi(z) \in \overline{\mathbb{Q}}(z)$, $\deg \varphi \geq 2$. The Fatou set $\mathcal{F}$ is the largest open set on which the iterates of φ form a normal family and the Julia set $\mathcal{J}$ is its complement.

Example: $\varphi(z) = z^2$, $\mathcal{J} = S^1$, $\mathcal{F} = \mathbb{P}^1(\mathbb{C}) \setminus S^1$. Two attracting basins make up the Fatou set, one around 0 and one around ∞.

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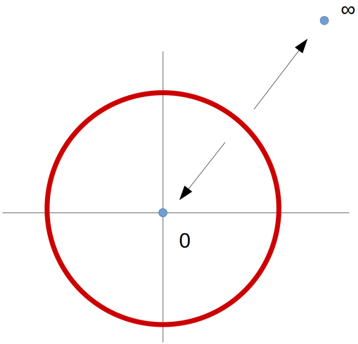

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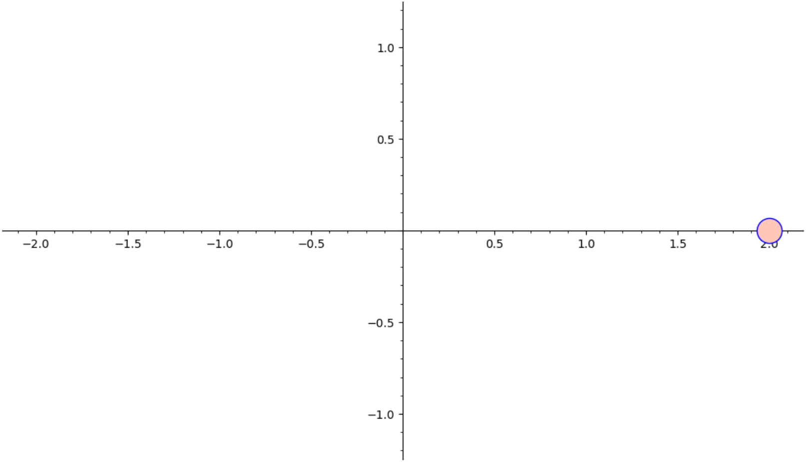

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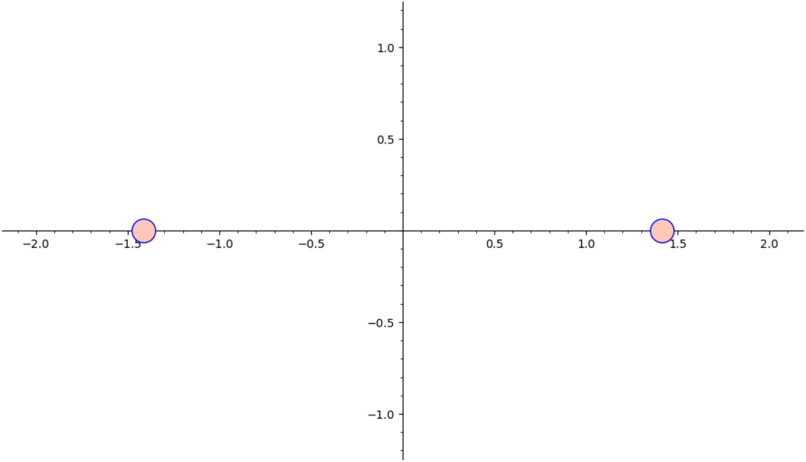

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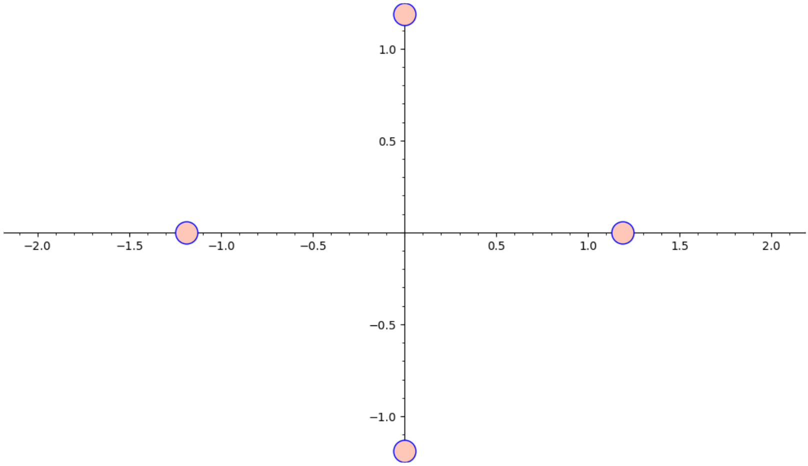

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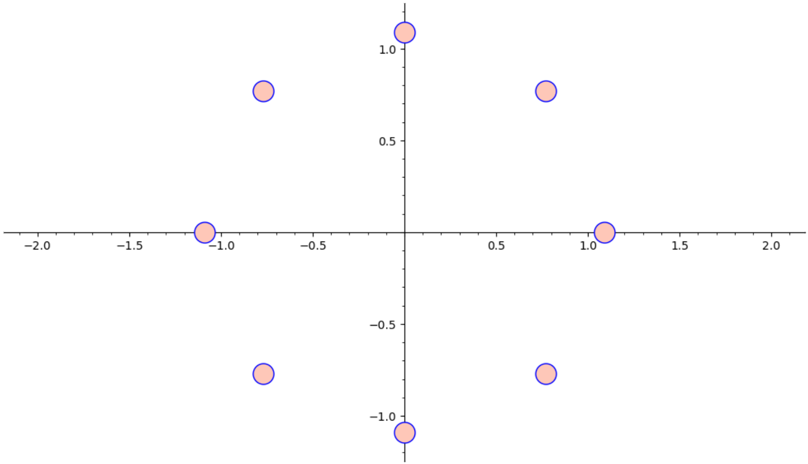

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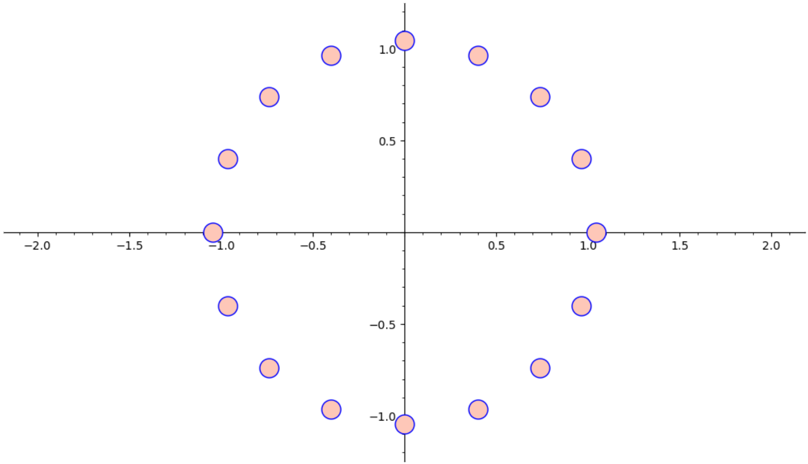

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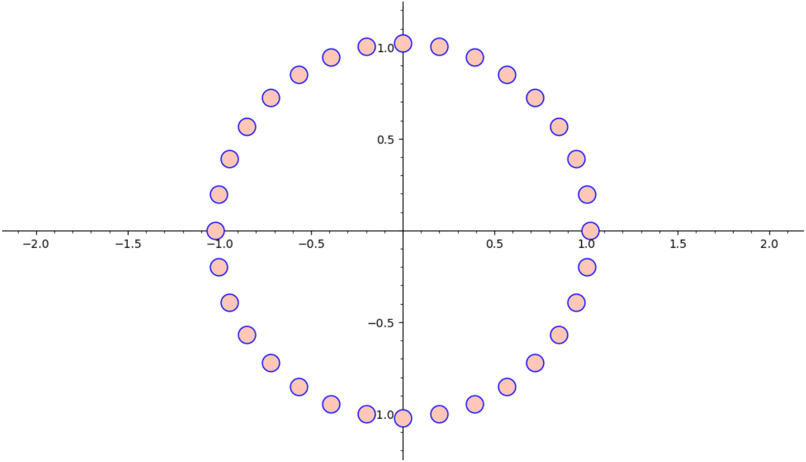

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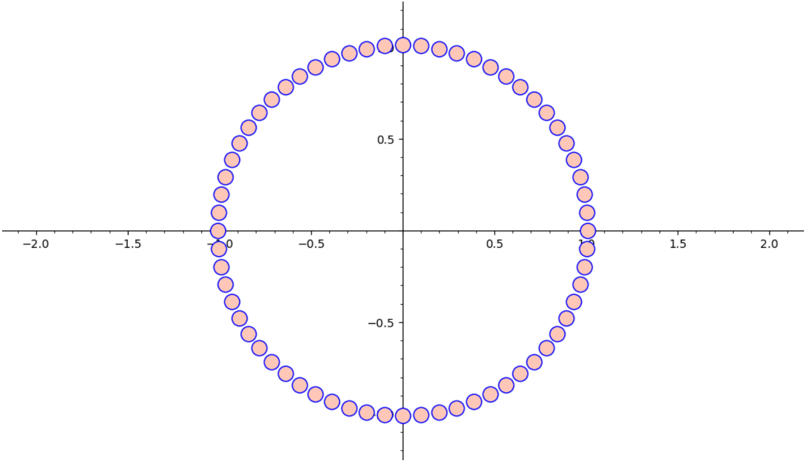

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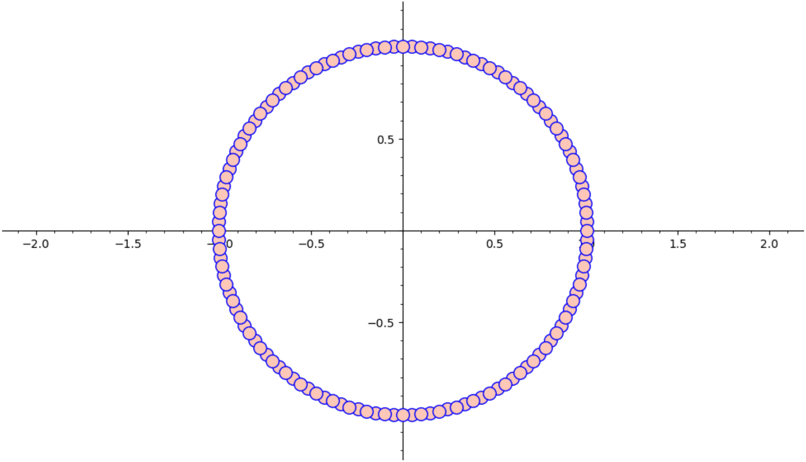

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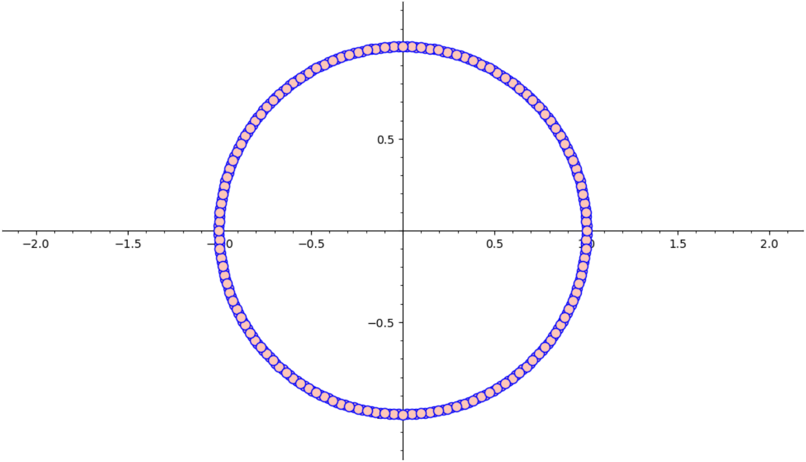

Equidistribution theorems in arithmetic dynamics

Let $\varphi(z) \in K(z)$, $d = \deg \varphi \geq 2$, for a number field $K/\mathbb{Q}$.

We define the dynamical height $h_\varphi(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} h(\varphi^n(\alpha))$.

The dynamical height is the height associated to an adelic measure: for every place $v \in M_K$, there exists a probability measure $\rho_{\varphi,v}$ on the (Berkovich) projective line $\mathrm{P}^1(\mathbb{C}_v)$ such that $h_\varphi = h_{\rho_\varphi}$.

Equidistribution theorem (Baker-Rumely, Favre-Rivera-Letelier '06, et al.): If α_n is a sequence of distinct points in $\mathbb{P}^1(\overline{K})$ such that $h_\varphi(\alpha_n) \to 0$, then the probability measures

$$\frac{1}{[K(\alpha) : K]} \sum_{z \in \mathrm{Gal}(\overline{K}/K)\cdot\alpha} \delta_z \xrightarrow{w} \rho_{\varphi,v}.$$

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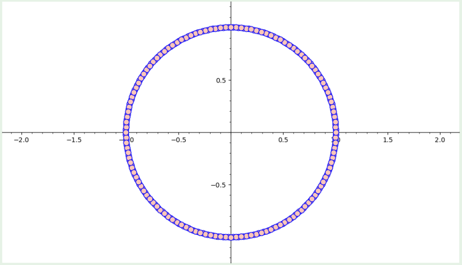
(Bilu '97) If $\varphi(z) = z^2$, then $h_\varphi = h$ and

$$\rho_{\varphi,\infty} = \lambda_\infty = \frac{d\theta}{2\pi}\Big|_{S^1}$$



As an example, $h(2^{1/n}) \to 0$, so for any $f \in C(\mathbb{P}^1(\mathbb{C}))$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(2^{1/n}e^{2\pi ik/n}) \to \int f(z)\, d\lambda_\infty(z) = \int_0^1 f(e^{2\pi it})\, dt$$

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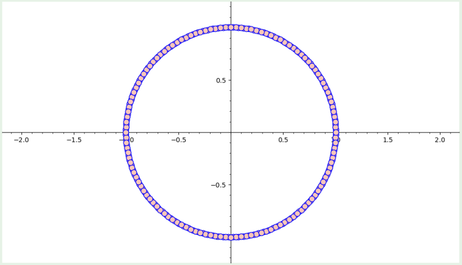

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Stochastic dynamical systems

Suppose we have a family $S \subset \overline{\mathbb{Q}}(z)$ of rational maps (always assume degree ≥ 2) together with a probability measure ν_1 on S. We think of $\nu_1(\varphi)$ as the likelihood of selecting the $\varphi \in S$ during a random walk. Let $\gamma_n = (\varphi_1, \ldots, \varphi_n) \in S^n$ act as $\gamma_n(\alpha) = \varphi_n \circ \cdots \circ \varphi_1(\alpha)$ and endow S^n with the product measure $\nu_n = \nu_1 \times \cdots \times \nu_1$.

Theorem (Healey-Hindes '19, cf. Kawaguchi '07)

If the φ are height controlled in the sense that the constants $C_\varphi = \sup_\alpha |(\deg \varphi)h(\alpha) - h(\varphi(\alpha))|$ are bounded, then

$$h_S(\alpha) = \lim_{n\to\infty} \mathop{\mathbb{E}}_{\gamma_n \in S^n} \frac{h(\gamma_n(\alpha))}{\deg \gamma_n}$$

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Compare stochastic height to the classical height:

$$h_S(\alpha) = \lim_{n\to\infty} \mathop{\mathbb{E}}_{\gamma_n \in S^n} \frac{h(\gamma_n(\alpha))}{\deg \gamma_n} \qquad h_\varphi(\alpha) = \lim_{n\to\infty} \frac{h(\varphi^n(\alpha))}{(\deg \varphi)^n}$$

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$$h_S(\alpha) = 0 \iff \mathcal{O}_S(\alpha) = \{\gamma_n(\alpha) : n \geq 0,\ \gamma_n \in S^n\} \text{ is finite.}$$

Unfortunately, $\{\alpha \in \mathbb{P}^1(\overline{K}) : h_S(\alpha) = 0\}$ is infinite

$\iff \bigcap_{\varphi \in S} \mathrm{PrePer}(\varphi)$ is infinite

$\iff \mathrm{PrePer}(\varphi) = \mathrm{PrePer}(\psi) \iff h_\varphi = h_\psi\ \forall \varphi, \psi \in S.$

(Assume throughout $\nu_1(\varphi) > 0\ \forall \varphi \in S$.)

Properties of the Stochastic Height

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Why so few points of low height?

Indeed, we have

$$\{h_S = 0\} \subset \bigcap_{\varphi \in S} \mathrm{PrePer}(\varphi)\,.$$

Let's consider an example with $S = \{f, g\}$ with $f(z) = z^2 - 2$ and $g(z) = z^2$. Notice that $\mathrm{PrePer}(f) \cap \mathrm{PrePer}(g) = \{-1, 0, 1, \infty\}$ but

$$\begin{tikzcd}
 & 4 \arrow[rr, dashed, "\text{limit under } f{,}g"] & & \infty \arrow[loop above, "f"] \arrow[loop below, "g"] & \\
0 \arrow[r, "g"] \arrow[loop below, "f"] & -2 \arrow[u, "f"] \arrow[r, "g"] & 2 \arrow[ul, "f"'] \arrow[loop below, "g"] & -1 \arrow[loop left, "g"] \arrow[r, bend left, "f"] & 1 \arrow[l, bend left, "g"] \arrow[loop right, "f"]
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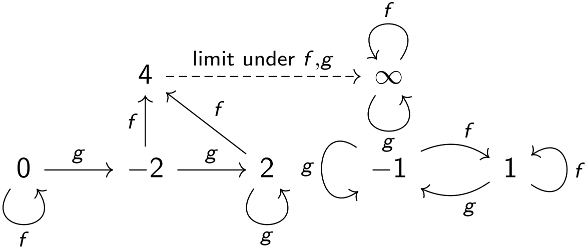

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So are stochastic dynamics interesting?

Yes. Hinkkanen and Martin in '96 introduced the study of dynamics for semigroups of rational functions and defined (in $\mathbb{P}^1(\mathbb{C})$) the Fatou and Julia sets of a semigroup, like that generated by S, but at first glance, the lack of an infinite set of height zero points would seem to preclude too many interesting equidistribution-type results, particularly those with an arithmetic flavor.

But what about equidistribution of preimages?

Let's consider an example, $S = \{f, g\}$ with $\nu_1(f) = \nu_1(g) = 1/2$ and $f(z) = z^2$ and $g(z) = 2z^2$. For a starting point $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$, how do we define 'preimages' here?

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From preimages to a random backwards orbit

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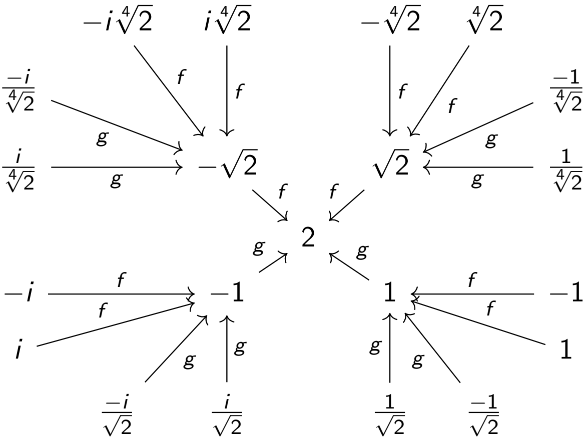

Random backwards orbit

Let $\Delta_{0,\alpha} = \delta_\alpha$. For $z \in \mathbb{P}^1$, let $m_\varphi(z)$ denotes the multiplicity of φ at z, then

$$\Delta_{1,\alpha} = \sum_{\varphi \in S} \sum_{z \in \varphi^{-1}(\alpha)} \frac{m_\varphi(z)}{\deg \varphi} \nu_1(\varphi) \delta_z = \mathbb{E}_S \sum_{z \in \varphi^{-1}(\alpha)} \frac{m_\varphi(z)}{\deg \varphi} \delta_z .$$

Note that we can define a pullback of measures for rational maps, which satisfies

$$\varphi^*(\delta_z) = \sum_{w \in \varphi^{-1}(z)} \frac{m_\varphi(w)}{\deg \varphi} \delta_w$$

then

$$\Delta_1 = \mathbb{E}_S \frac{\varphi^*(\Delta_0)}{\deg \varphi}, \quad \text{recursively define} \quad \Delta_{n+1} = \frac{\varphi^*(\Delta_n)}{\deg \varphi}.$$

Does $\Delta_{n,\alpha}$ equidistribute according to some measure? If so, what?

Do we need to worry about multiplicities?

We really need to worry about multiplicities as we take backwards orbits. Let's consider the example of the preimages of 0 under $\varphi(z) = z^2 - 1$.

$$
\begin{tikzcd}
 & & & 0 \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
 & & -1 \arrow[dl, Rightarrow, "2"'] & & +1 \arrow[dl, "1"'] \arrow[dr, "1"] & \\
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\end{tikzcd}
$$

with multiplicities:

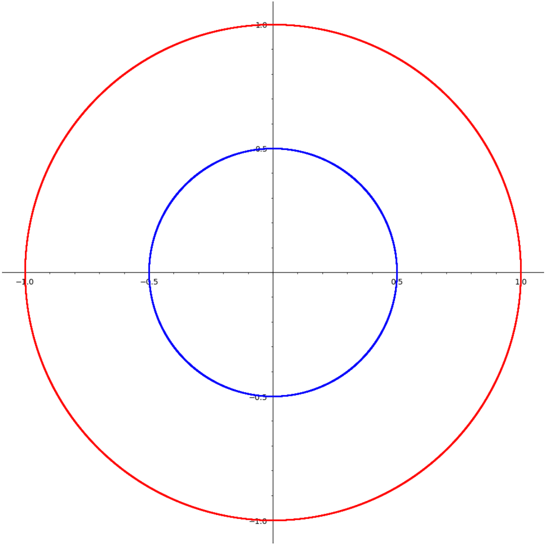
2 2 1 1 1 1

If we have multiple maps in our family S, then it is not even obvious *prima facie* that our limit measures won't charge individual points.

Distribution in our example

$S = \{f, g\}$, $\nu_1(f) = \nu_1(g) = 1/2$, $f(z) = z^2$ and $g(z) = 2z^2$.

What do the $\Delta_{n,\alpha}$ look like?



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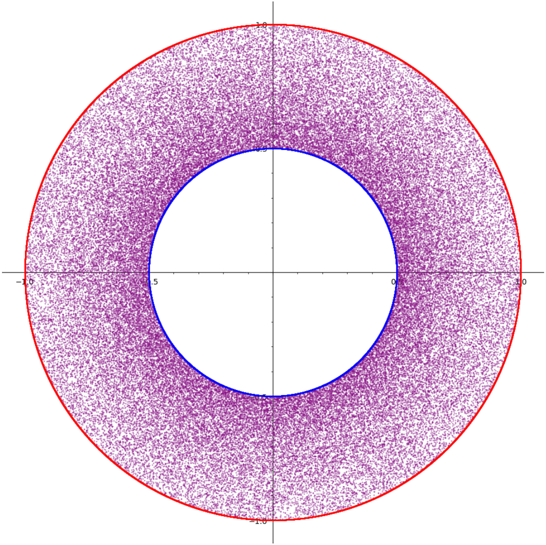

Was that a fluke? Let's try a less trivial example

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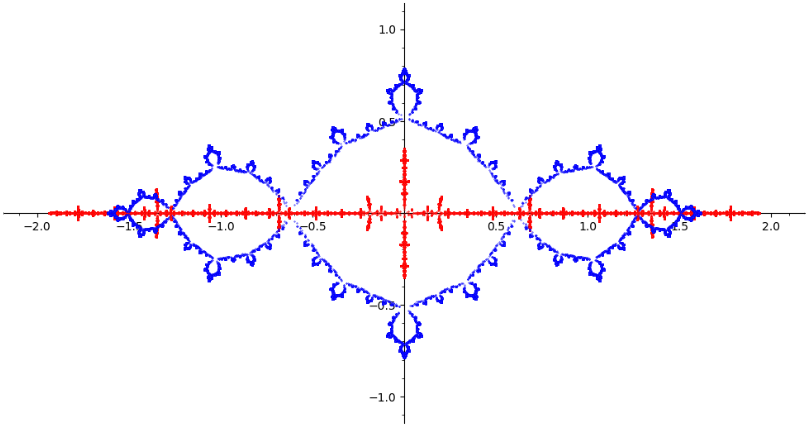

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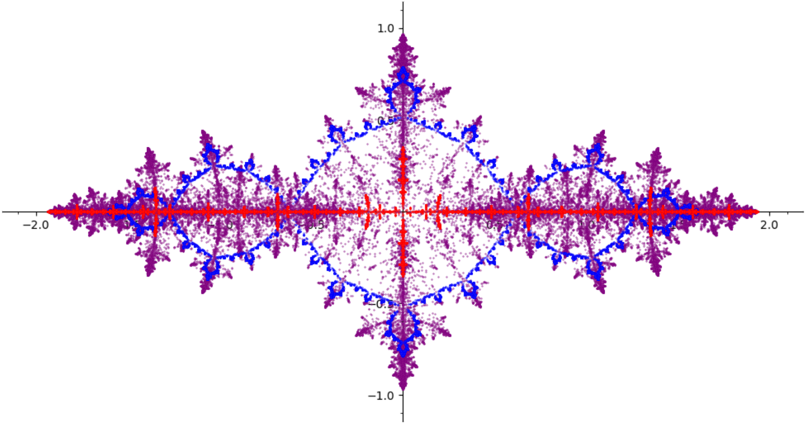

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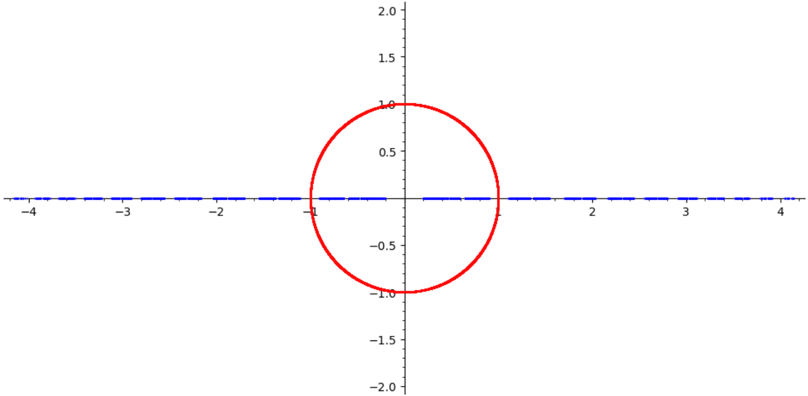

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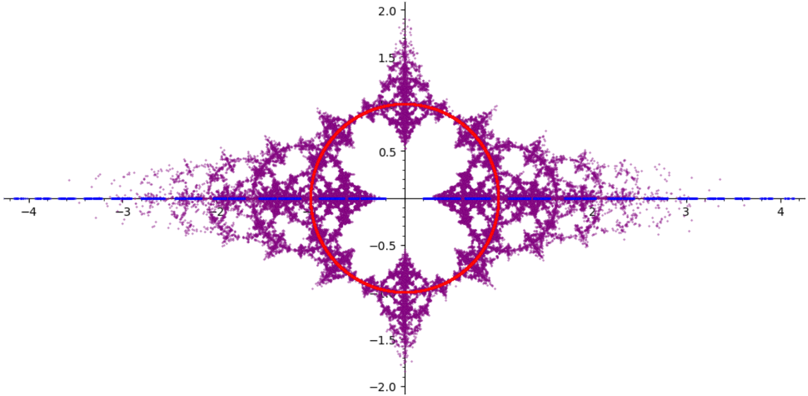

What about more than two functions?

$S = \{f, g, h\}$, $\nu_1(f) = 0.7, \nu_1(g) = 0.2, \nu_1(h) = 0.1$,
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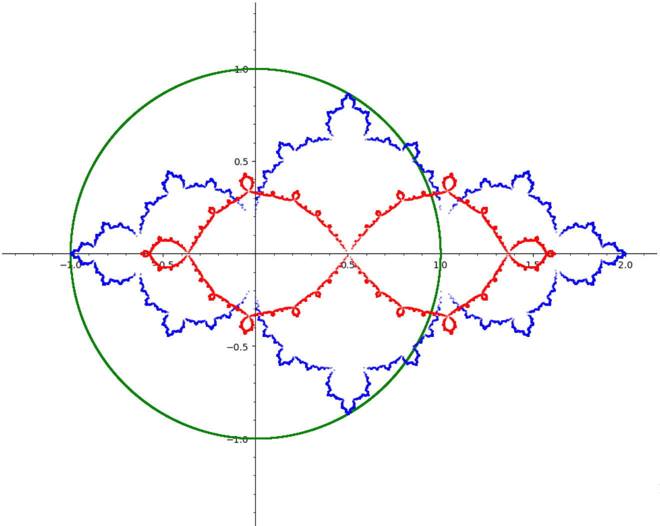

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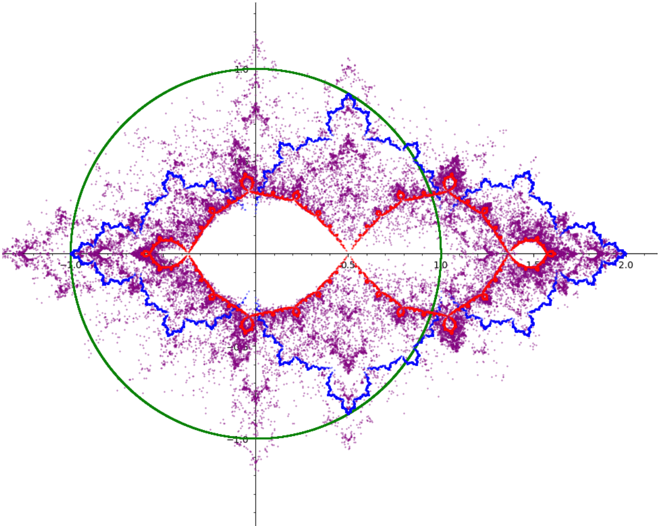

Shifting the odds?

$S = \{f, g, h\}$, $\nu_1(f) = 0.1, \nu_1(g) = 0.2, \nu_1(h) = 0.7$,
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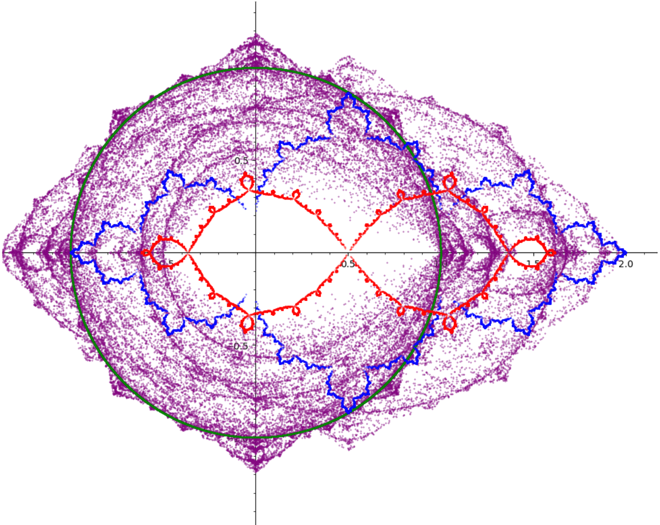

Ok, well, what about four maps?

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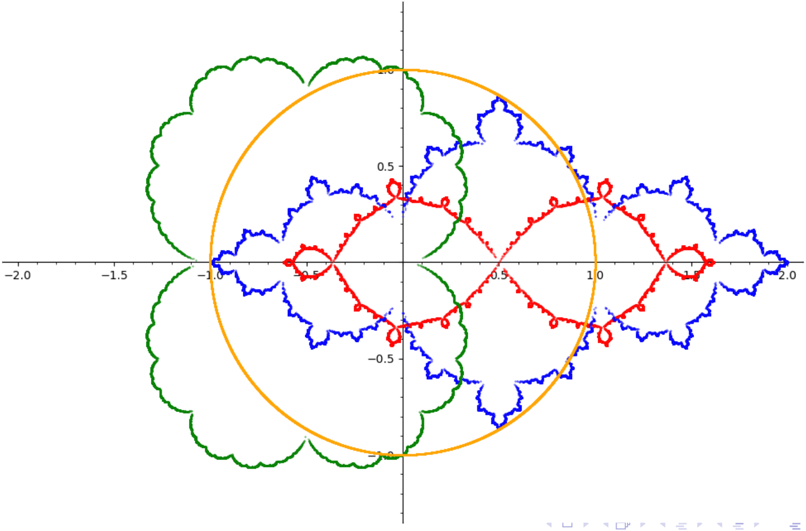

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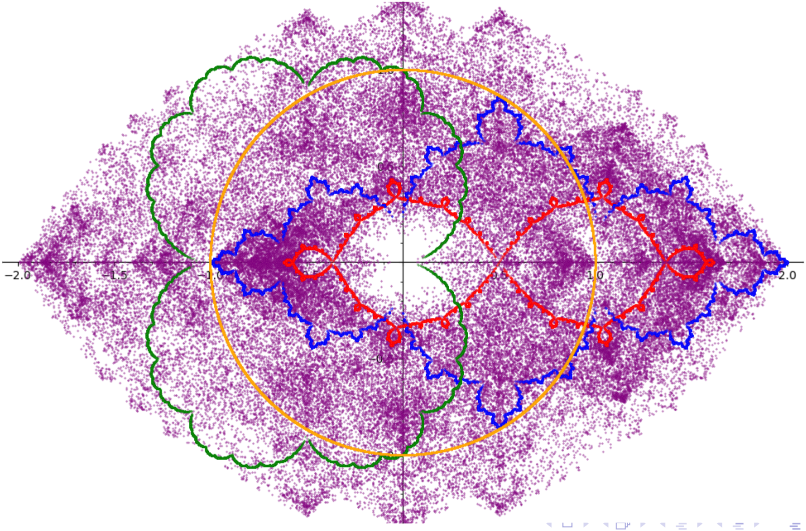

Alright, back to math...

Let's briefly recall the potential theoretic construction of the dynamical height. Say we have–

OK OK, maybe just ONE more...

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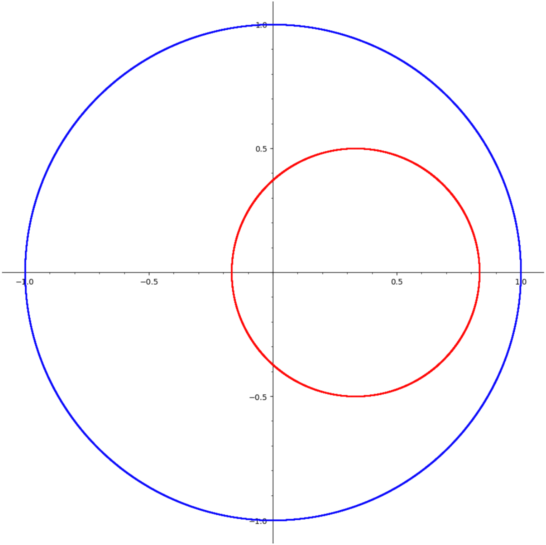
OK OK, maybe just ONE more...

I promise this is a math talk

$S = \{f, g\}$, $\nu_1(f) = \nu_1(g) = 1/2$, $f(z) = 2(z - 1/3)^2 + 1/3$ and $g(z) = z^2$. What do the $\Delta_{n,\alpha}$ look like?



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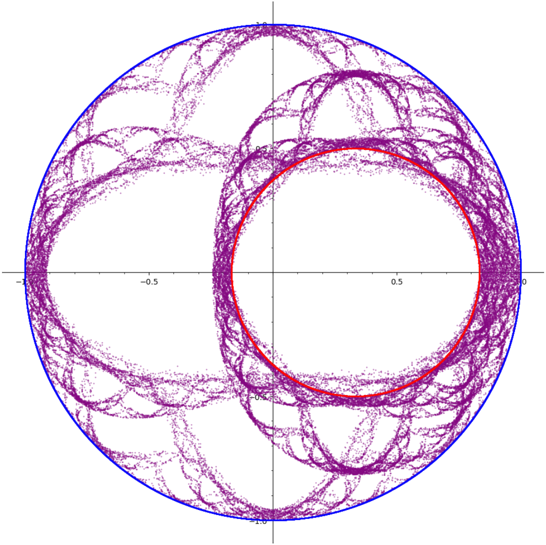

So what should we be *proving* here?

These pictures are highly suggestive that there really is something nontrivial to prove here. If our set S is finite and everything is defined over a number field K, then:

- There is an adelic measure $\rho = (\rho_v)_{v \in M_K}$ and an associated height function h_ρ such that $h_S = h_\rho$.
- Random backwards orbits are equidistributing according to ρ at every place of K. In some sense, $h_S(\Delta_{n,\alpha}) \to 0$.

Note the Healey-Hindes paper extended the results of Kawaguchi in two directions: it allowed the family S to be infinite, and it allowed for different probability measures ν_1 on S. Both still require everything to be defined over a single base number field for most results. What if for S infinite, we want to allow the field of definition to grow as well? Note that for S infinite, the backwards orbit $\Delta_{n,\alpha}$ has infinite support even for $n = 1$!

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The main new construction: Generalized adelic measures

In order to allow for an infinite family $S \subset \overline{\mathbb{Q}}(z)$ without assuming the field of definition is of finite degree over the rationals we need extend the current way we work with heights. A brief overview of the inspiration:

1. Favre and Rivera-Letelier '06 introduced the notion of adelic measures. These have a finite number of 'bad' places, where they local height is allowed to differ from the standard height, but it must admit a continuous potential with respect to the standard measure at each place. They proved an extremely general equidistribution theorem for these heights.
2. Mavraki and Ye '17, while studying parametrized families of rational maps, discovered that the associated heights failed to be adelic: they differed from the standard measure at infinitely many places; but in a controlled fashion. They introduced the notion of a quasi-adelic measure, and proved these were still Weil heights and one could prove equidistribution.

The main new construction: Generalized adelic measures

We introduced the notion of a generalized adelic measure, and proved the standard results: our measures define Weil heights, and we proved an equidistribution theorem for points of low height. In fact, our proof works not just for heights of single algebraic numbers, but for arbitrary discrete probability measures on $\mathbb{P}^1(\overline{\mathbb{Q}})$, definining a notion first for when such measures have finite height. We then proved that:

Theorem (Doyle, F., Tobin '21)

Suppose $S \subset \overline{\mathbb{Q}}(z)$ and ν_1 is a probability measure on S and the family is L^1 height controlled. Then there exists a generalized adelic measure ρ_S such that $h_S = h_{\rho_S}$. Further, for almost every place y of $\overline{\mathbb{Q}}$, and any $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ which is not exceptional for S, the random backwards orbit measures $\Delta_{n,\alpha}$ equidistribute according to $\rho_{S,y}$.

Let's briefly recall the potential theoretic construction of the dynamical height. Say we have a rational map $\varphi(z) = f(z)/g(z) \in K(z)$. For each $v \in M_K$ the standard measure is given on $\mathrm{P}^1(\mathbb{C}_v)$ by the equilibrium measure of unit disc:

$$\lambda_v = \begin{cases} \left.\dfrac{d\theta}{2\pi}\right|_{S^1} & \text{if } v \mid \infty \\ \delta_{\zeta_{0,1}} & \text{if } v \nmid \infty \end{cases}$$

When φ has good reduction at v, $\rho_{\varphi,v} = \lambda_v$.

Recall the standard height is

$$h(\alpha) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\alpha|_v \quad \text{and} \quad \Delta \log^+ |z|_v = \lambda_v$$

Background on heights and adelic measures

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Recall the standard height is (take Laplacian on the whole line)

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Then if $\varphi(z) = f(z)/g(z)$, we let

$$g_{1,v}(z) = \frac{1}{\deg \varphi} \log \max\{|f(z)|_v, |g(z)|_v\} - \log^+ |z|_v + C$$

satisfies $\Delta g_{1,v} = \dfrac{\varphi^*(\lambda_v)}{\deg \varphi} - \lambda_v$. We can get further pullbacks using the formula:

$$\Delta(g \circ \varphi) = \varphi^*(\Delta g)$$

For the dynamical height associated to φ, this leads to the telescoping series construction:

$$g_v(z) = \sum_{n=0}^{\infty} \frac{g_{1,v} \circ \varphi^n}{(\deg \varphi)^n} \quad \text{yields} \quad \Delta g_v = \rho_{\varphi,v} - \lambda_v.$$

Notice that $g_v : \mathrm{P}^1(\mathbb{C}_v) \to \mathbb{R}$ is continuous.

We say $(\rho_v)_{v\in M_K}$ is an adelic measure if

1. For all but finitely many v, $\rho_v = \lambda_v$ is the standard measure.
2. For each v there is a continuous $g_v : \mathrm{P}^1(\mathbb{C}_v) \to \mathbb{R}$ with $\Delta g_v = \rho_v - \lambda_v$.

If we define the energy pairing

$$(\rho,\sigma)_v = \iint_{\mathrm{A}^1_v \times \mathrm{A}^1_v \setminus \mathrm{Diag}_v} -\log |z-w|_v \, d\rho(z)\, d\sigma(w)$$

then we can define $[\alpha]_K = \frac{1}{\# G_K\cdot\alpha}\sum_{z\in G_K\cdot\alpha}\delta_z$ and

$$h_\rho(\alpha) = \frac{1}{2}\sum_{v\in M_K} N_v(\rho_v - [\alpha]_K, \rho_v - [\alpha]_K)_v$$

where $N_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$.

How can we extend for an infinite degree base field?

Gubler in 1997 and later Allcock and Vaaler in 2009 introduced a way to write the height as an *integral over all places of* $\overline{\mathbb{Q}}$ for a certain measure which naturally keeps track of the 'local over global degree' factors which normalize the height.

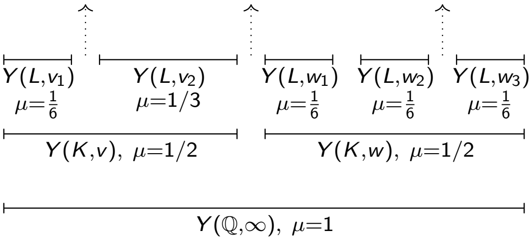
The fundamental idea is similar to that used in forming the absolute Galois group: a place of $\overline{\mathbb{Q}}$ is actually an element of the projective limit of the space of places for all number fields $K \subset \overline{\mathbb{Q}}$ partially ordered by inclusion. Over any single prime p of $\mathbb{Q}$, the set of places in a number field K lying over p, $M_{K,p}$, is finite:

$$Y(\mathbb{Q}, p) = \varprojlim_{K/\mathbb{Q}} M_{K,p}$$

Then $Y = \bigcup_{p \in M_{\mathbb{Q}}} Y(\mathbb{Q}, p)$. Let $Y(K, v) = \{y \in Y : y \mid v\}$.

Measure μ on the places of $\overline{\mathbb{Q}}$

Let $K = \mathbb{Q}(\alpha)$, $\alpha^2 - 2 = 0$, and $L = K(\beta)$ a cubic extension of K generated by $\beta^3 + 2\alpha\beta - 1 = 0$. How does the place ∞ of $\mathbb{Q}$ split in K and L?



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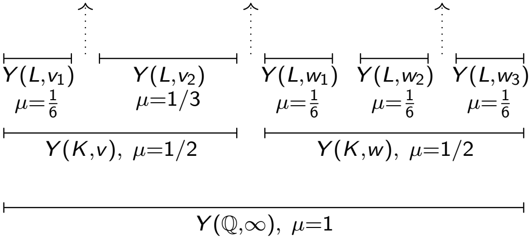

Two places of K, corresponding to $v : \alpha \mapsto \sqrt{2}$ and $w : \alpha \mapsto -\sqrt{2}$.

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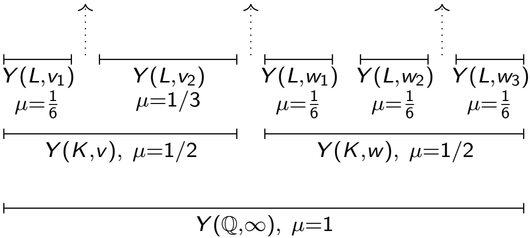
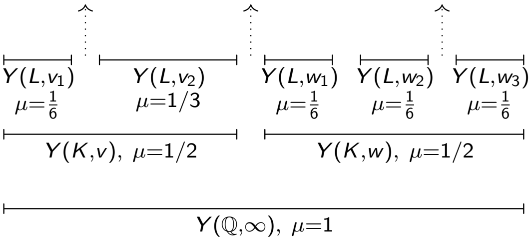


Two places of K, corresponding to $v : \alpha \mapsto \sqrt{2}$ and $w : \alpha \mapsto -\sqrt{2}$. Note that β has 4 real roots and 1 complex conjugate pair of roots. The complex pair of roots occurs over v.

Measure μ on the places of $\overline{\mathbb{Q}}$



$$\mu(Y(K,v)) = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} = \frac{1}{2}, \text{ while}$$

$$\mu(Y(L,v_2)) = \frac{[L_{v_2} : \mathbb{Q}_{v_2}]}{[L : \mathbb{Q}]} = \frac{[\mathbb{C} : \mathbb{R}]}{[L : \mathbb{Q}]} = \frac{2}{6} = \frac{1}{6}.$$

In general, the extension formula guarantees:

$$\sum_{\substack{v \in M_K \\ v \mid p}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} = 1.$$

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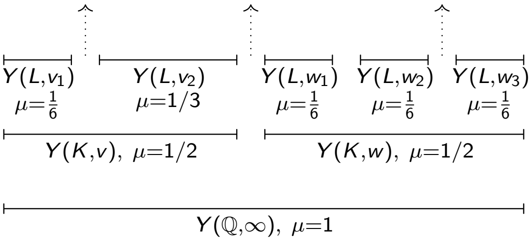

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Height as an integral over Y

$$h(\alpha) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ |\alpha|_v = \int_Y \log^+ |\alpha|_y \, d\mu(y).$$

The height for a (generalized) adelic measure ρ then becomes:

$$h_\rho(\alpha) = \frac{1}{2} \int_Y (\rho_y - \delta_\alpha, \rho_y - \delta_\alpha)_y \, d\mu(y).$$

Or, more generally, for a discrete probability measure Δ on $\mathbb{P}^1(\overline{\mathbb{Q}})$,

$$h_\rho(\alpha) = \frac{1}{2} \int_Y (\rho_y - \Delta, \rho_y - \Delta)_y \, d\mu(y).$$

So... what *is* a generalized adelic measure?

We say that $(\rho_y)_{y \in Y}$ is a generalized adelic measure if:

1. (Continuous potentials almost everywhere) For every rational prime p, there is a measurable function $g : Y(\mathbb{Q}, p) \times \mathrm{P}^1(\mathbb{C}_p) \to \mathbb{R}$ such that for μ-a.e. y, $g_y(z) = g(y, z) : \mathrm{P}^1(\mathbb{C}_y) \to \mathbb{R}$ is continuous and $\Delta g_y(z) = \rho_y - \lambda_y$, and
2. (L^1 height controlled) For every $y \in Y$, we normalize so that $g_y(\infty) = 0$ and let

$$C(y) = \sup_{z \in \mathrm{P}^1(\mathbb{C}_v)} |g_y(z)|$$

or $C(y) = \infty$ if we don't admit a continuous potential, then $C \in L^1(Y)$.

If ρ is defined over a single number field, then this is equivalent to being a quasi-adelic measure.

We prove the following:

Theorem (Doyle, F., Tobin '21)

Let ρ be a generalized adelic measure. Then h_ρ is an essentially positive Weil height, that is, $h_\rho = h + O(1)$ and $\{h_\rho < -\epsilon\}$ is finite for every $\epsilon > 0$.

Theorem (Doyle, F., Tobin '21)

If Δ_n is a sequence of discrete probability measures on $\mathbb{P}^1(\overline{\mathbb{Q}})$ such that $h_\rho(\Delta_n) \to 0$ and Δ_n is well-distributed, then for μ-almost every place y, we have $\Delta_n \to \rho_y$ in the sense of weak convergence of measures.

We say Δ_n is well-distributed if $\sum_{z \in \mathbb{P}^1(\overline{\mathbb{Q}})} \Delta_n(z)^2 \to 0$. This replaces conditions like 'the degrees of α_n tend to infinity.'

GAMs associated to stochastic families

We prove that if (S, ν_1) is L^1 height controlled, then $h_S = h_{\rho,S}$ where we define ρ_S via:

$$\Delta g_{\varphi,y} = \frac{\varphi^*(\lambda_y)}{\deg \varphi} - \lambda_y \qquad C_\varphi(y) = \sup_z |g_{\varphi,y}(z)|$$

$$g_{1,y}(z) = \mathbb{E}_S g_{\varphi,y}(z) = \sum_{\varphi \in S} \nu_1(\varphi) g_{\varphi,y}(z),$$

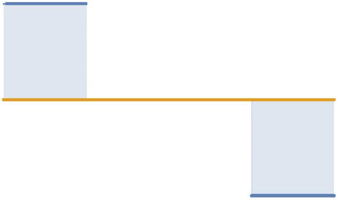
$$g_{n+1,y}(z) = \mathbb{E}_{\gamma_n \in S^n} \frac{g_{1,y}(\gamma_n(z))}{\deg(\gamma_n)}.$$

$$g_{S,y}(z) = \sum_{n=1}^{\infty} g_{n,y}(z) \quad \text{and} \quad \Delta g_{S,y} = \rho_{S,y} - \lambda_y.$$

The family is L^1-height controlled if $\mathbb{E}_S C_\varphi(y) \in L^1(y)$.

An example of a generalized but not quasi-adelic measure

Let $\tau_n \searrow \theta_0$ be Salem numbers, where θ_0 is the positive root of $x^3 - x - 1$, with rapidly rising degree, say $[\mathbb{Q}(\tau_n) : \mathbb{Q}] \geq 3^n$. Let $\alpha_n = \tau_n^{2[\mathbb{Q}(\tau_n):\mathbb{Q}]}$, $S = \{\varphi_n(z) = \alpha_n z^2 : n \in \mathbb{N}\}$, and $\nu_1(\varphi_n) = 1/2^n$.
The idea: what do the valuations of τ look like over ∞?



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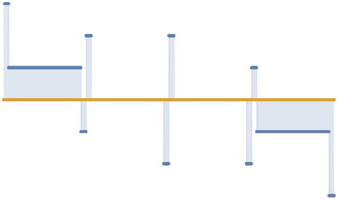

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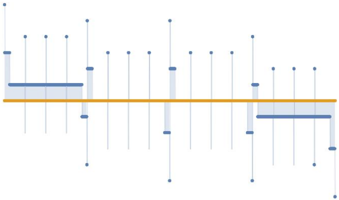

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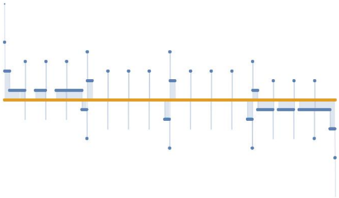

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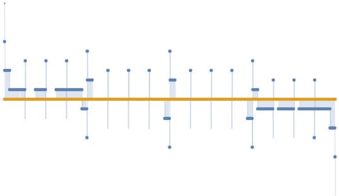
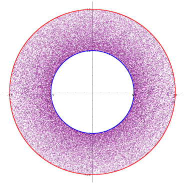


The Borel-Cantelli lemma says that for μ-a.e. $y \in Y(\mathbb{Q}, \infty)$, at most finitely $|\alpha_n|_y \neq 1$. One can show (S, ν_1) is L^1 height controlled and defines a generalized adelic measure.

Back to our initial example

Recall our initial example:

$S = \{f, g\}$, $\nu_1(f) = \nu_1(g) = 1/2$, $f(z) = z^2$ and $g(z) = 2z^2$. What is ρ_S?



$$d\rho_{S,\infty}(re^{i\theta}) = \begin{cases} \dfrac{dr}{r\log 2}\dfrac{d\theta}{2\pi} & \text{if } 1/2 \le r \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$J_{S,\infty} = \{1/2 \le |z| \le 1\}$$

While $J_{S,2} = [\zeta_{0,1}, \zeta_{0,2}]$ with

$$d\rho_{S,2}(x) = \begin{cases} \dfrac{dx}{x\log 2} & \text{if } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

While $\rho_{S,p} = \lambda_p$ for all $p \neq 2, \infty$.

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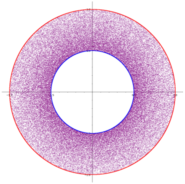

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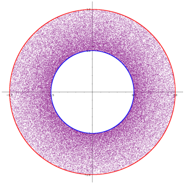
While $\rho_{S,p} = \lambda_p$ for all $p \neq 2, \infty$.

Recall our initial example:

$S=\{f,g\}$, $\nu_1(f)=\nu_1(g)=1/2$, $f(z)=z^2$ and $g(z)=2z^2$. What is ρ_S?



$$d\rho_{S,\infty}(re^{i\theta})=\begin{cases}\dfrac{dr}{r\log 2}\dfrac{d\theta}{2\pi} & \text{if } 1/2\le r\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

$$J_{S,\infty}=\{1/2\le |z|\le 1\}$$

While $J_{S,2}=[\zeta_{0,1},\zeta_{0,2}]$ with

$$d\rho_{S,2}(x)=\begin{cases}\dfrac{dx}{x\log 2} & \text{if } 1\le x\le 2,\\ 0 & \text{otherwise.}\end{cases}$$

While $\rho_{S,p}=\lambda_p$ for all $p\neq 2,\infty$.

Thank you!

Thank you all for listening, and thanks to the organizers, Khoa, Tom, and Jason, for organizing this conference!

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Thank you all for listening, and thanks to the organizers, Khoa, Tom, and Jason, for organizing this conference!

Ok, I can't help it, one more: $S = \{z + z^2, z - z^2\}$, $\nu_1(f) = \nu_1(g)$