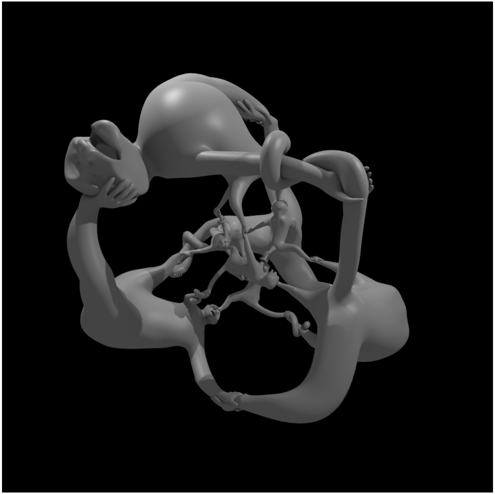


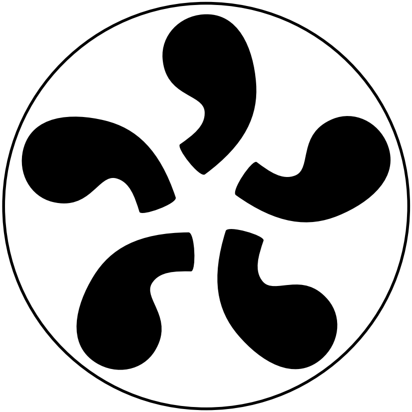
Henry Segerman
Oklahoma State University

Vi Hart
Communications Design Group

The quaternion group as a symmetry group

Symmetry

A symmetry of an object is a motion that leaves the object looking the same.

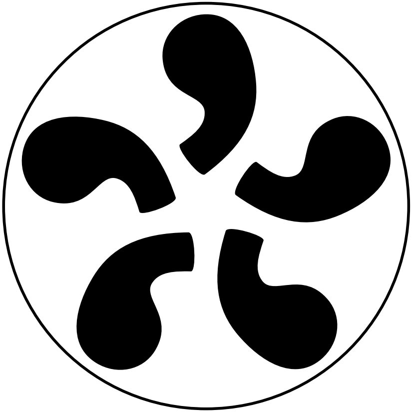


Symmetry

A symmetry of an object is a motion that leaves the object looking the same.

This object has five symmetries:

- Rotate by 1/5 of a turn,
- Rotate by 2/5 of a turn,
- Rotate by 3/5 of a turn,
- Rotate by 4/5 of a turn, and
- Do nothing.



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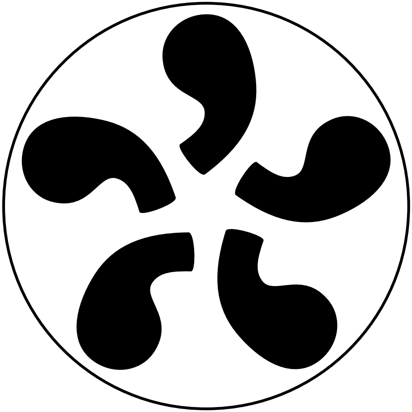

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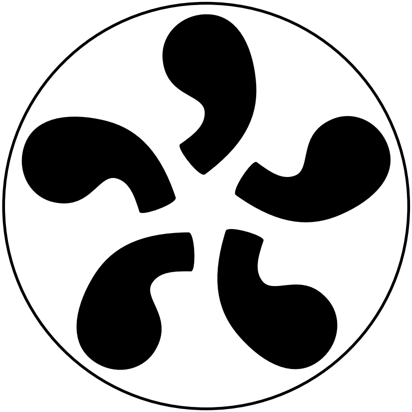
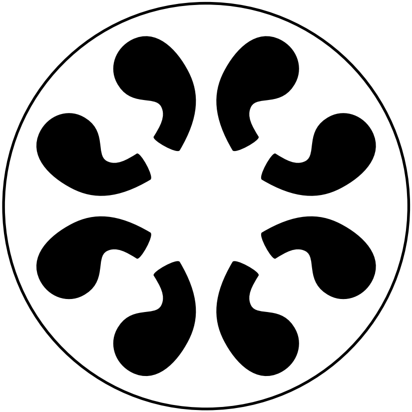


These symmetries can be "added together" by doing one motion followed by another.

The set of symmetries of an object is an example of a group, in this case $C_5 = \mathbb{Z}/5\mathbb{Z}$.

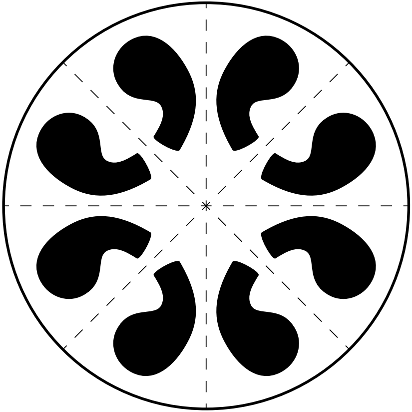
Symmetry



Symmetry

This object has eight symmetries:

- Four rotations (including do nothing), and
- Four reflections.



Symmetry

This object has eight symmetries:

- Four rotations (including do nothing), and
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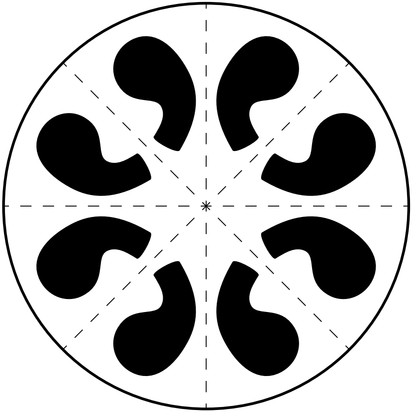
The symmetry group is D_4.



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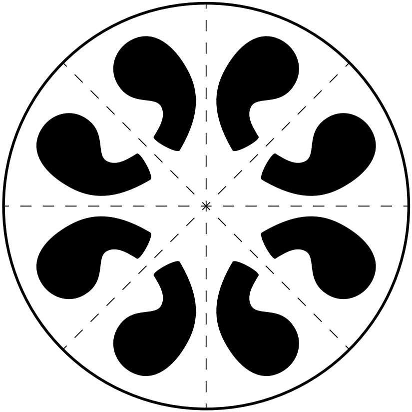

Groups are often studied and enumerated as abstract objects, independent of being the symmetry group of an object.

1. Which groups can be represented as the group of symmetries of some real-world physical object?

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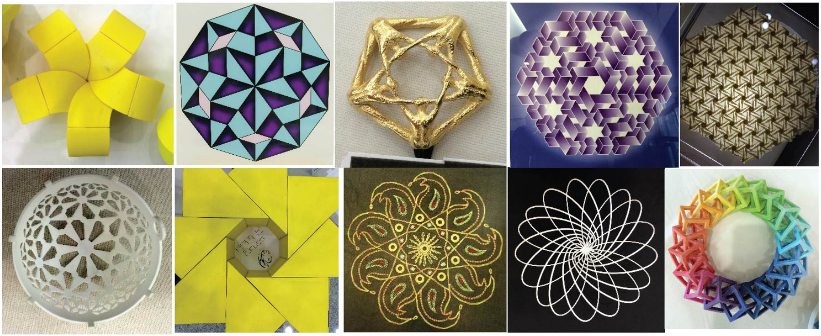
3. Which groups have been represented as the group of symmetries of some real-world physical object *at Bridges 2014*?

Bridges 2014 Symmetries: Bilateral ($C_2 = \mathbb{Z}/2\mathbb{Z}$)

Bridges 2014 Symmetries: Cyclic ($C_n = \mathbb{Z}/n\mathbb{Z}$), $n \leq 4$

Bridges 2014 Symmetries: Cyclic ($C_n = \mathbb{Z}/n\mathbb{Z}$), $n > 4$



What is the largest symmetry group represented at Bridges 2014?

What is the largest (finite) symmetry group represented at Bridges 2014?

The largest group award goes to Mike Naylor, for D_{360}

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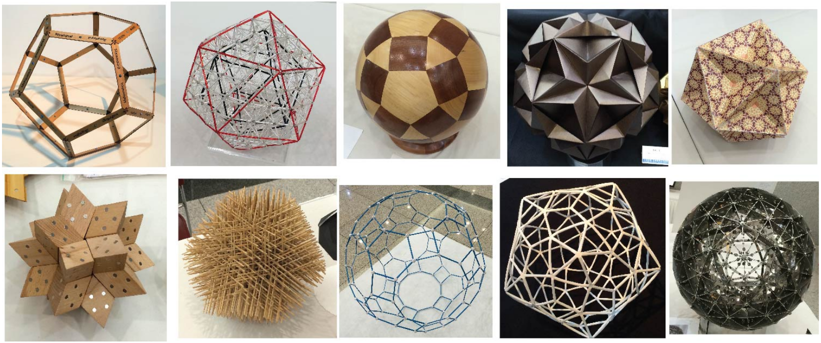
Unless we count the four-dimensional symmetry group of the 120-cell, more on this later...

Bridges 2014 Symmetries: Dihedral (D_n)

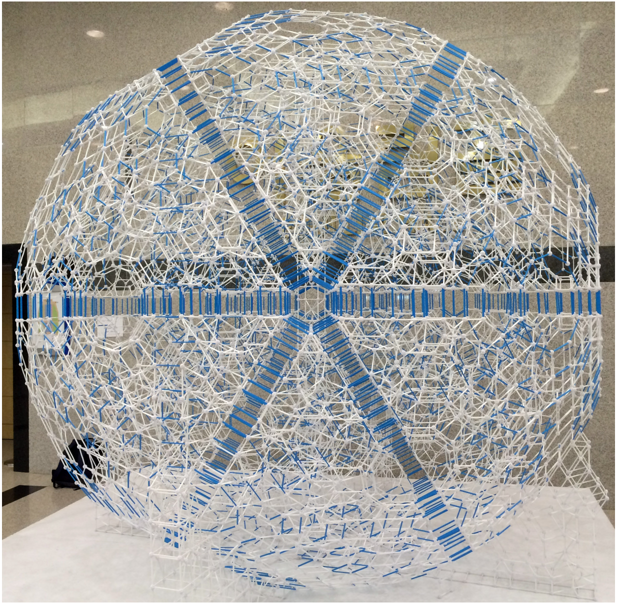
Which symmetry group is most represented at Bridges 2014?

Most popular group award goes to the dodecahedral reflection group

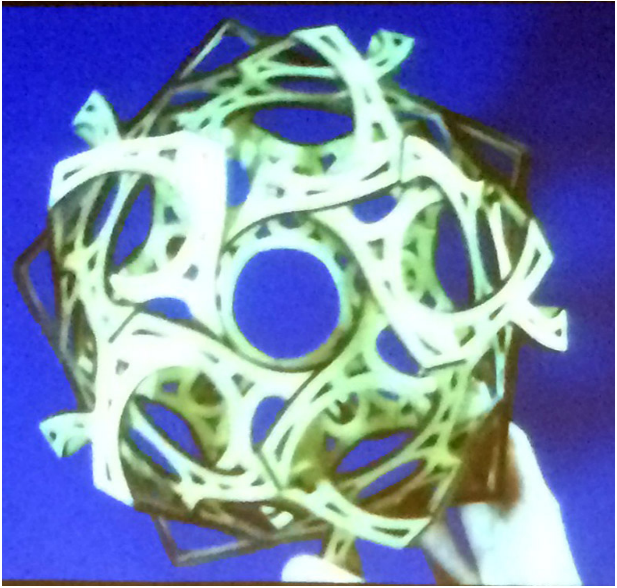


Most popular group award goes to the dodecahedral reflection group (the dodecahedral rotation group also makes a strong showing)

Yet another dodecahedral reflection group



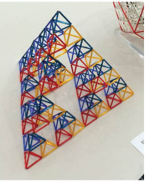
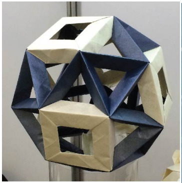
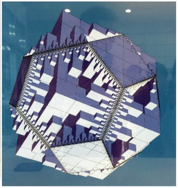
Yet another dodecahedral rotation group



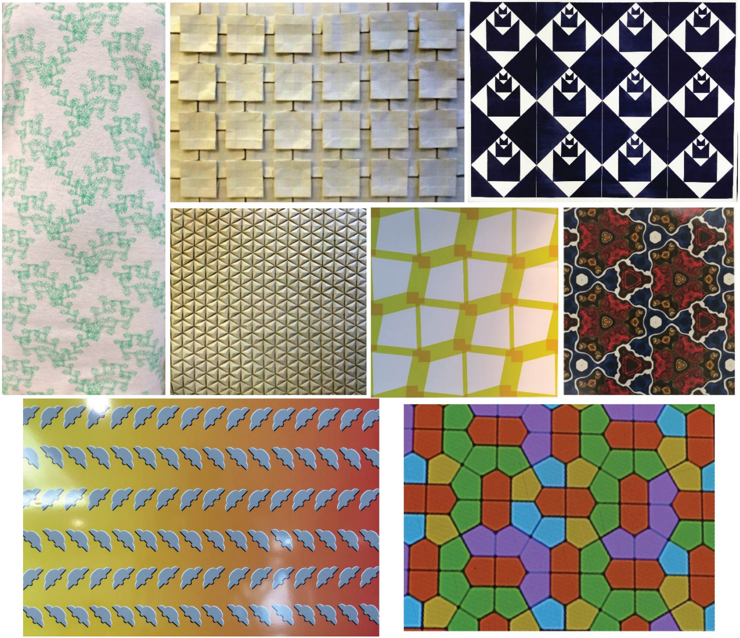
Bridges 2014 Symmetries: Other polyhedral groups






Bridges 2014 Symmetries: Wallpaper groups

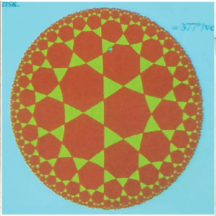


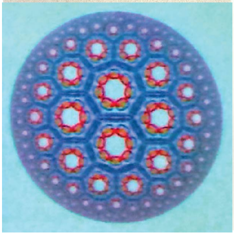
Bridges 2014 Symmetries: Miscellaneous groups



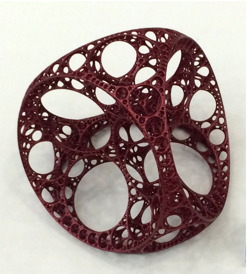


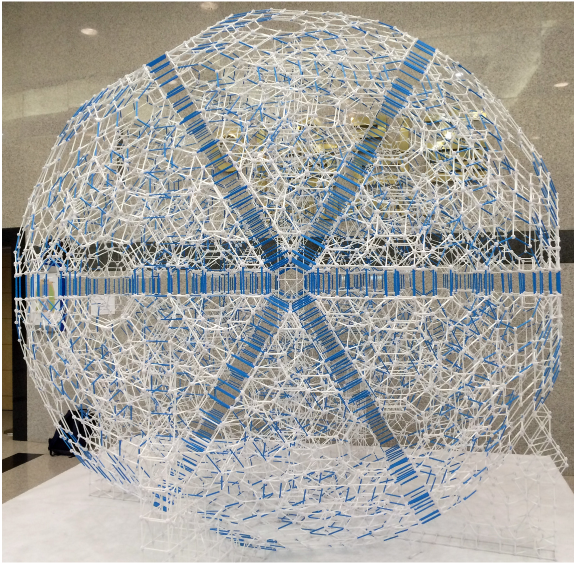

Bridges 2014 Symmetries: Hyperbolic?





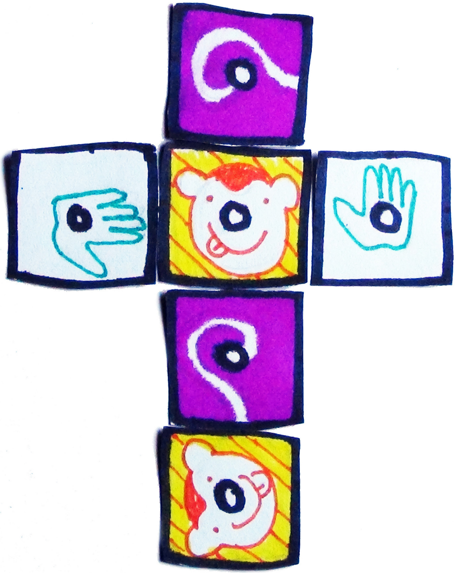
Bridges 2014 Symmetries: Four-dimensional?

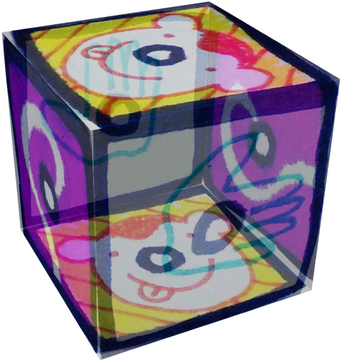




Are there any glaring gaps – small groups that should have a physical representation in a symmetric object but do not?

Monkey blocks





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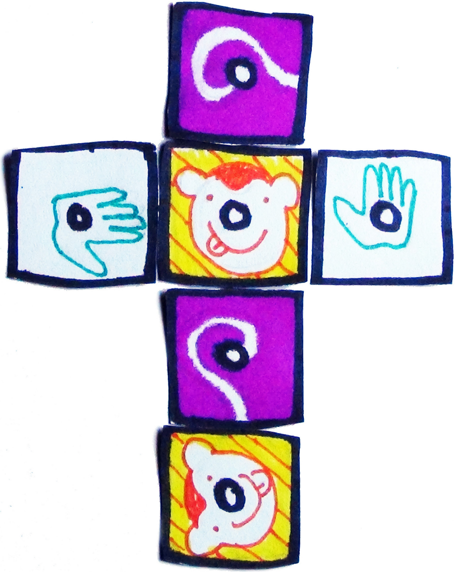

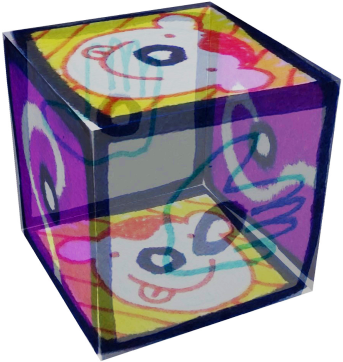

How can monkey blocks fit together so that the faces match?

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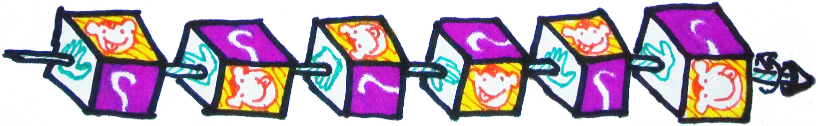

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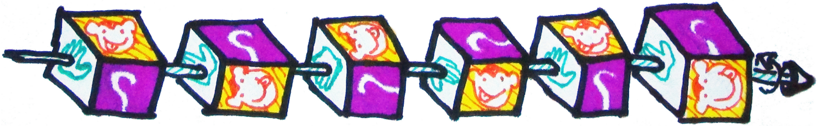

What are the symmetries of this infinite line of blocks?

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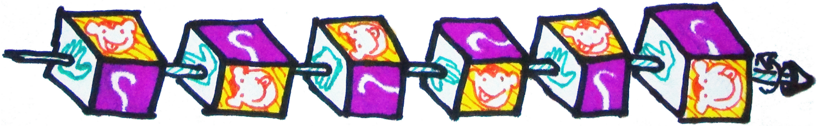

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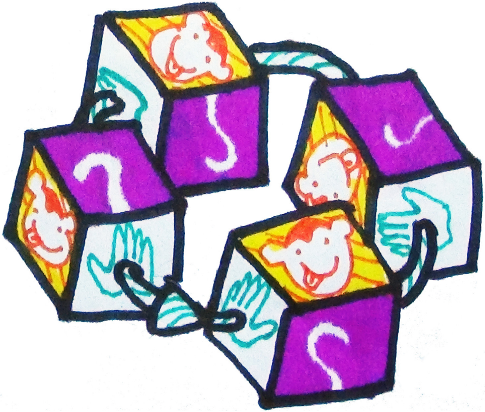

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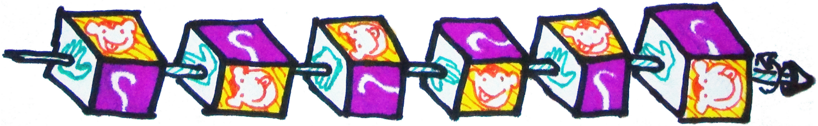
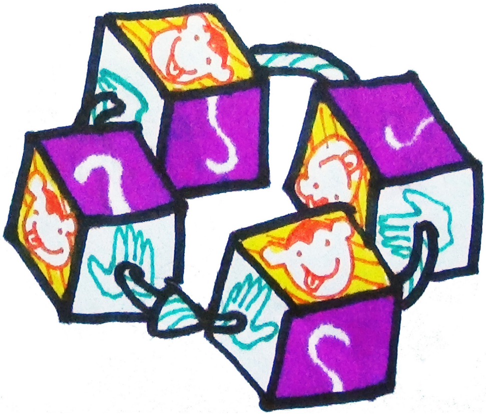


What are the symmetries of this infinite line of blocks?

What are the symmetries of this ring of four blocks?



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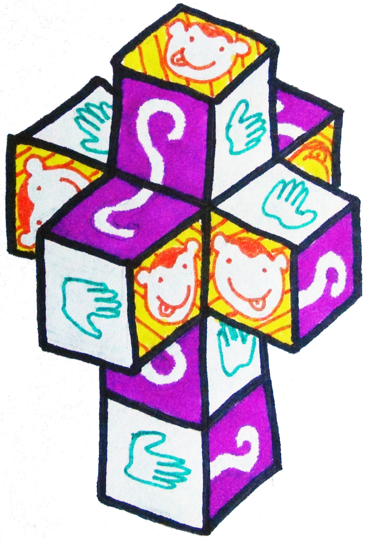

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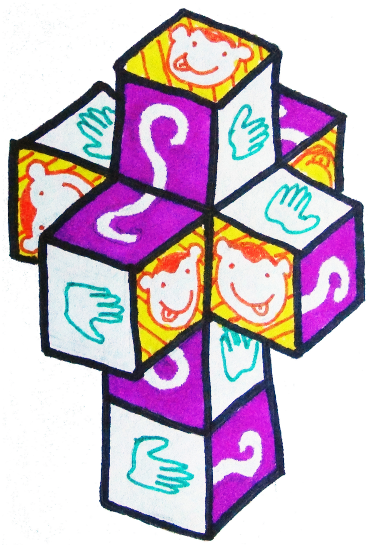

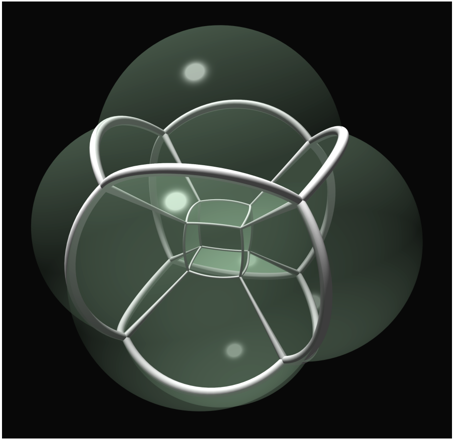

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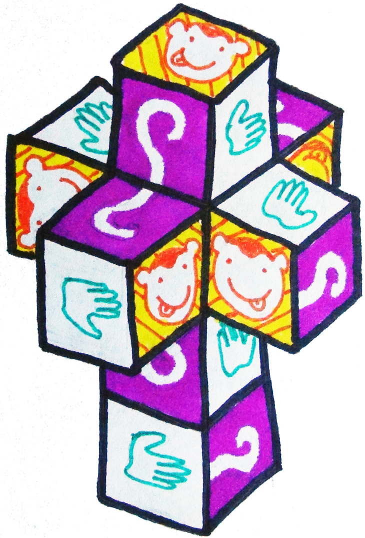
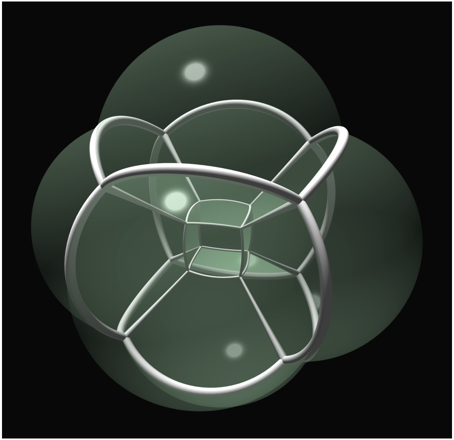
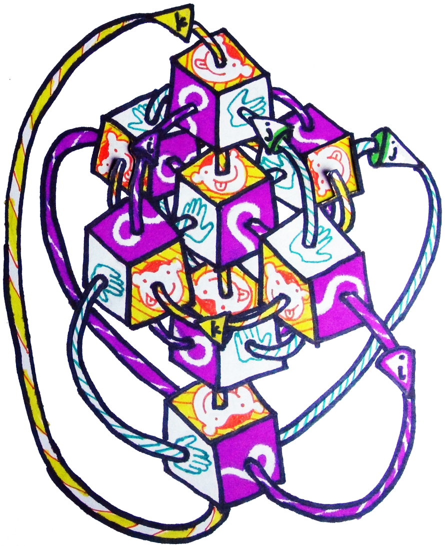


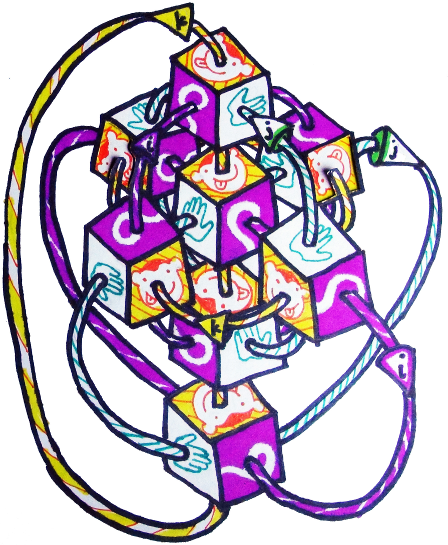


Eight monkey blocks glue together to make the cells of a hypercube!

How can monkey blocks fit together so that the faces match?



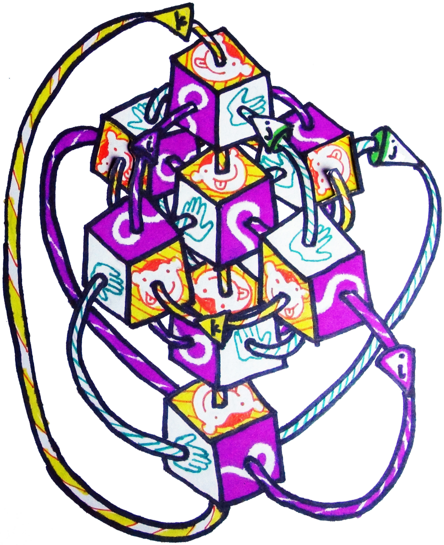
The quaternion group



There are eight symmetries of this decorated hypercube. These correspond to the eight elements of the quaternion group

$$Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$$

The quaternion group



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$$Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$$

- 1 is "do nothing",
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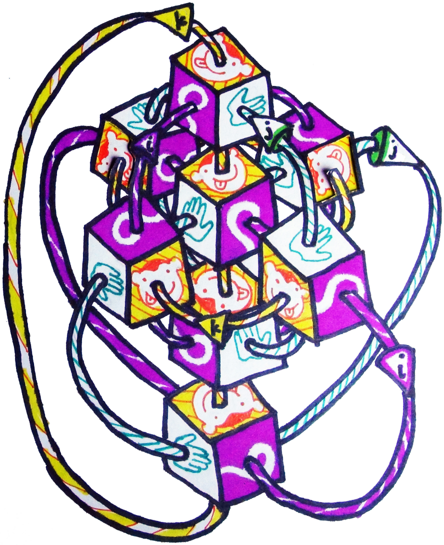

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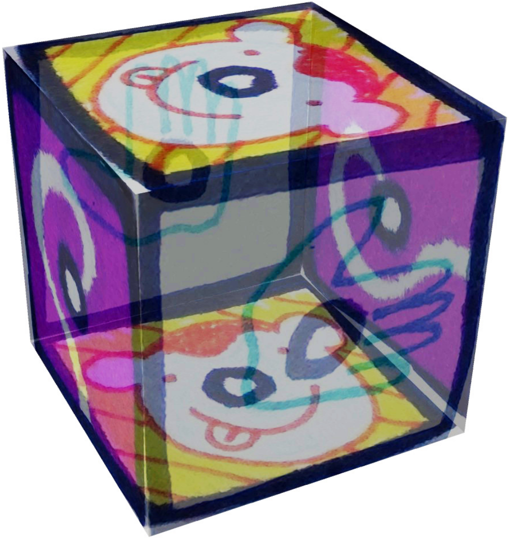
These satisfy
$i^2 = j^2 = k^2 = ijk = -1$.

A sculpture with Q_8 symmetry

Each monkey block itself has no symmetry.



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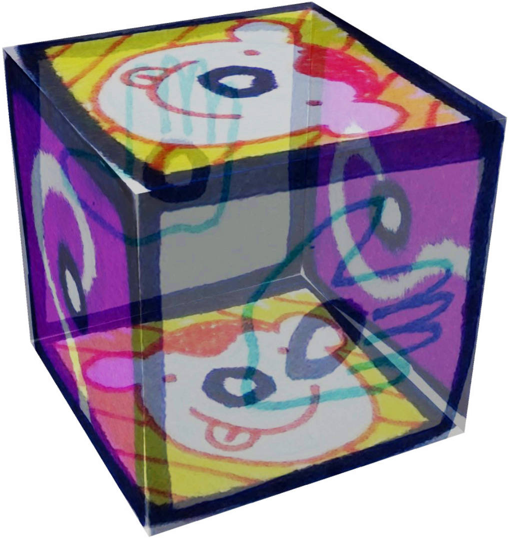

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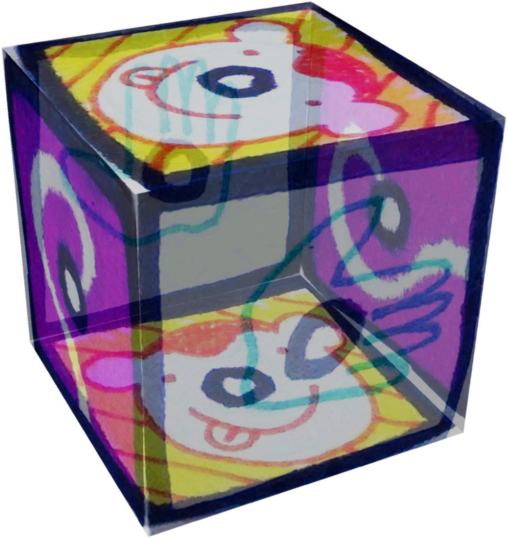
So to make a sculpture with Q_8 symmetry, we put a design with no symmetry into a cube.



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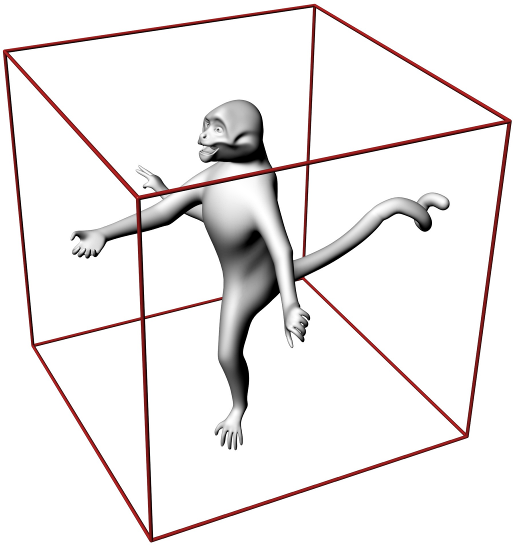

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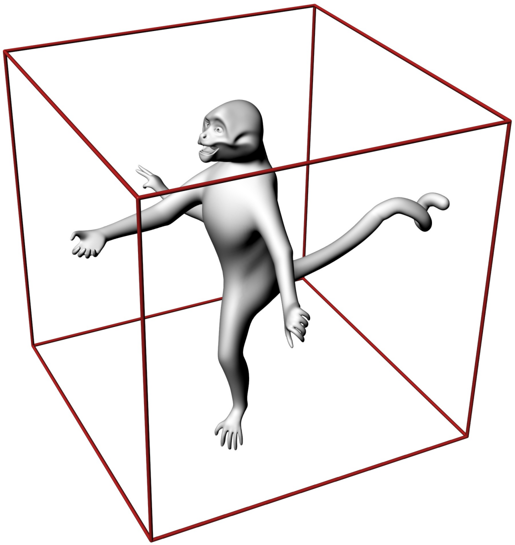

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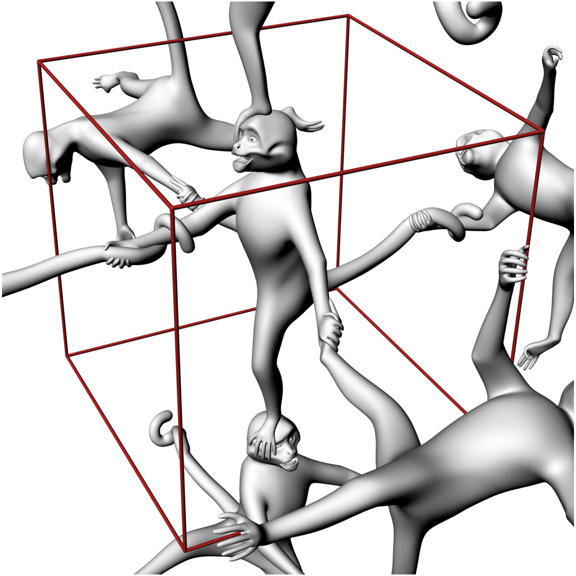
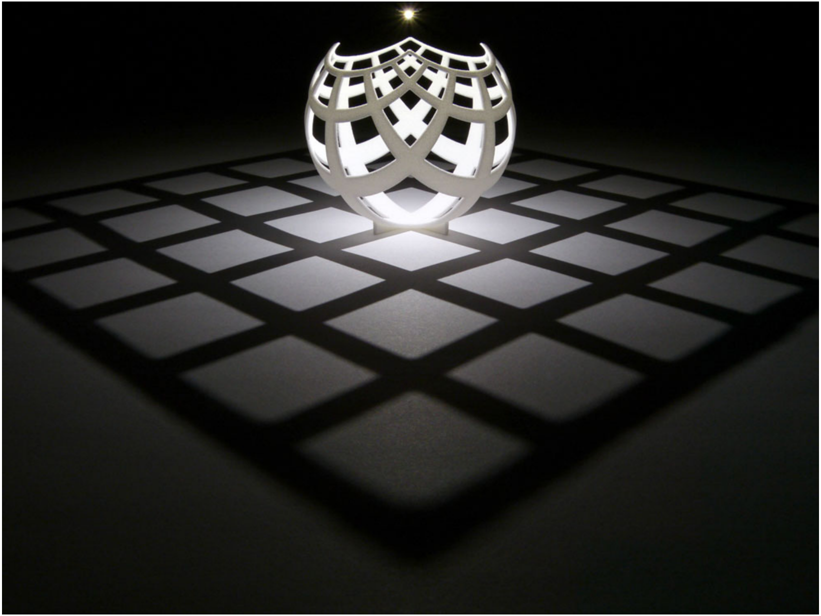
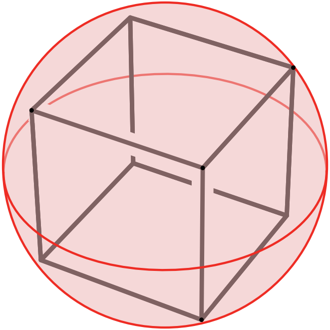
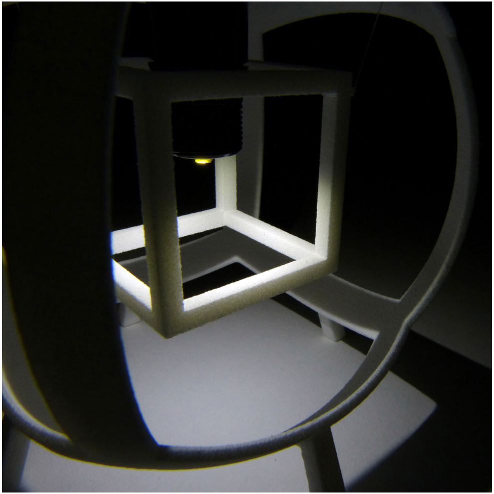


Drawing 4D pictures in 3D using stereographic projection



In 3D: First radially project the cube to the sphere...





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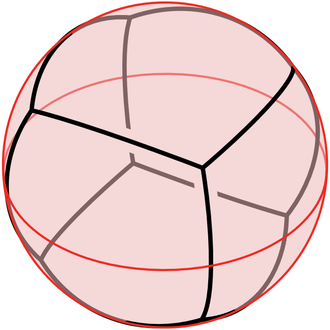
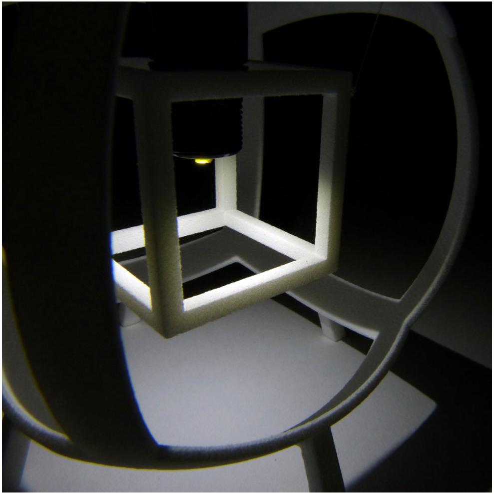
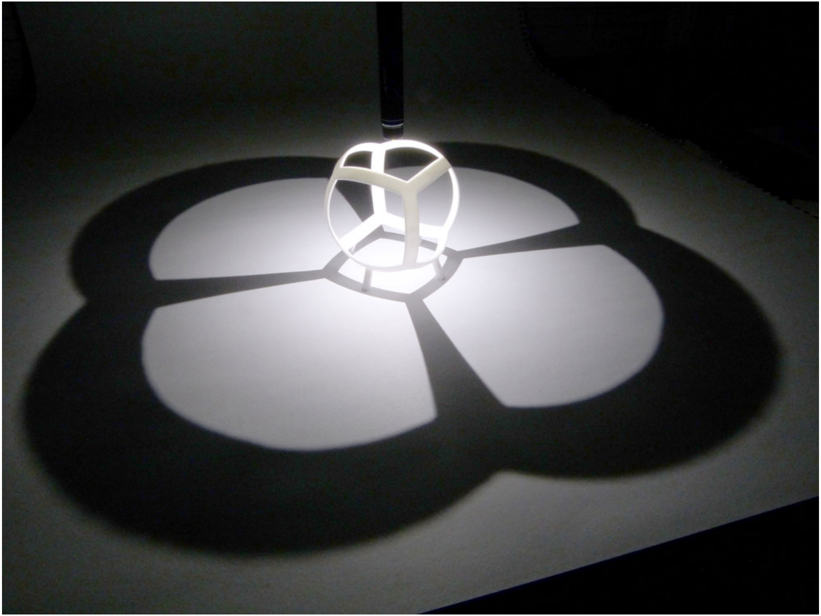




Then stereographically project to the plane



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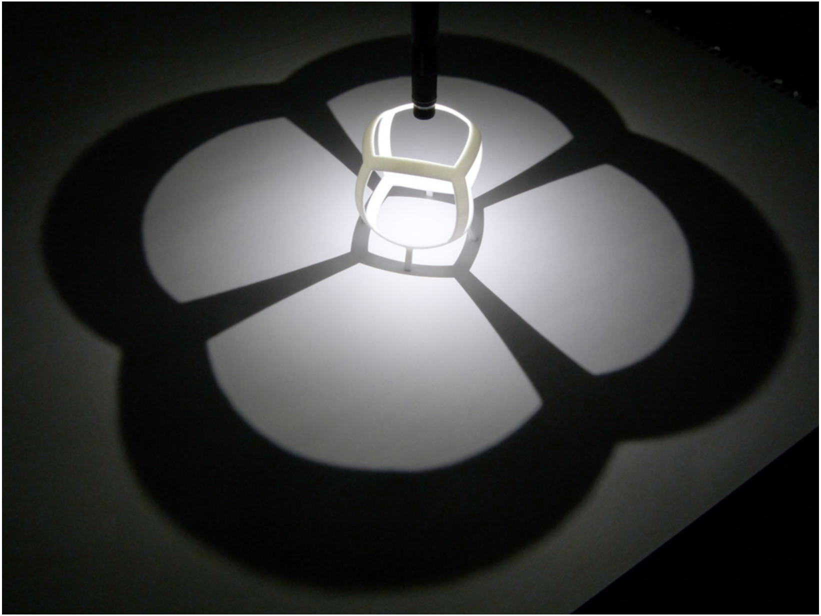
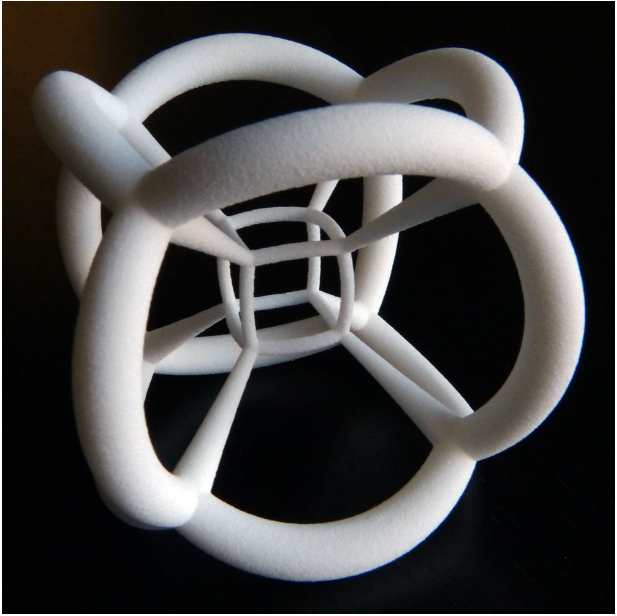


Do the same thing one dimension up to see a hypercube



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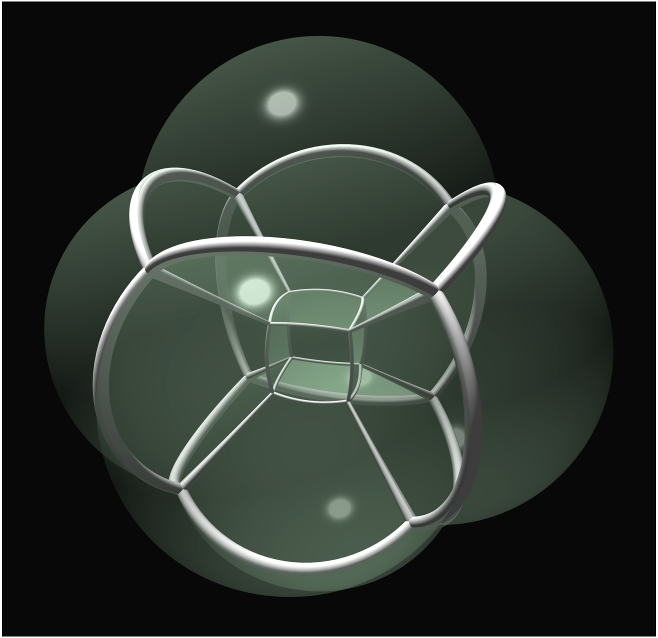
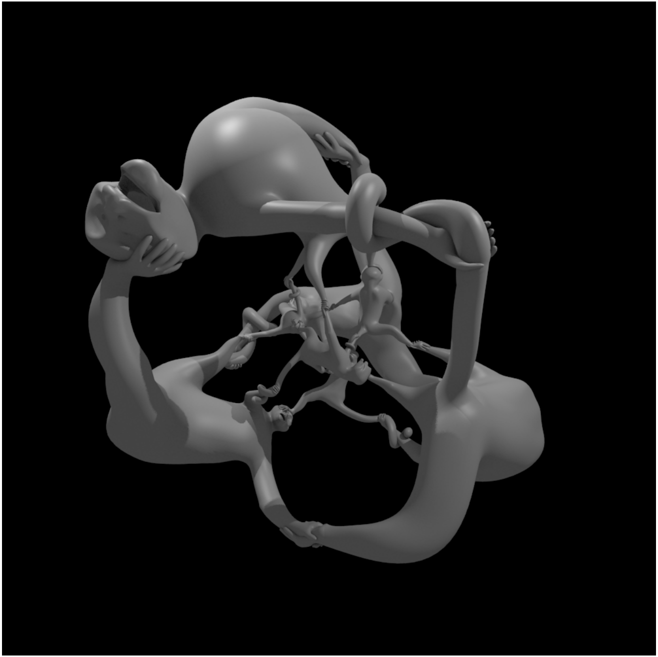
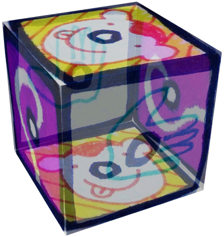


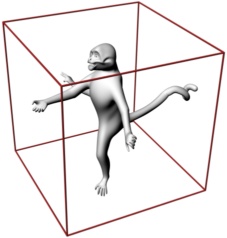
Do the same with the monkey design

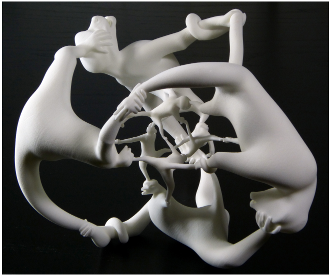
More fun than a hypercube of monkeys, by Henry Segerman and Will Segerman.



Thanks!







`segerman.org`

`math.okstate.edu/~segerman/`

`youtube.com/henryseg`

`shapeways.com/shops/henryseg`