Structure on the set of triangulations

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Acknowledgement: Many of the ideas in this talk have come from discussions with Craig D. Hodgson and J. Hyam Rubinstein.

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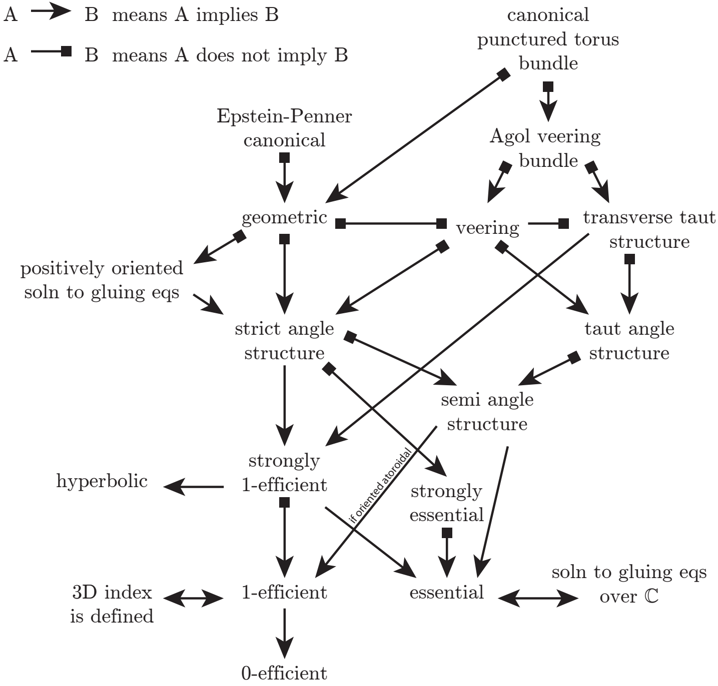
We are interested in finding triangulations with “good” properties.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Agol veering bundle
geometric
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transverse taut structure
positively oriented soln to gluing eqs
strict angle structure
taut angle structure
semi angle structure
strongly 1-efficient
hyperbolic
if oriented atoroidal
strongly essential
3D index is defined
1-efficient
essential
soln to gluing eqs over ℂ
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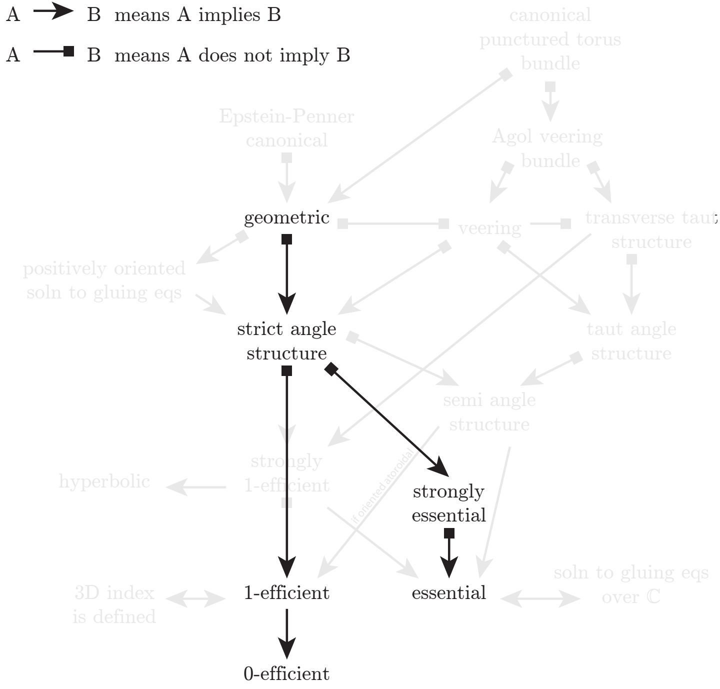
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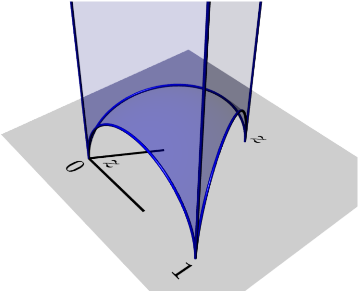
Geometric

A triangulation is geometric if the tetrahedra can be given ideal hyperbolic shapes that fit together to give the complete hyperbolic structure on the manifold.



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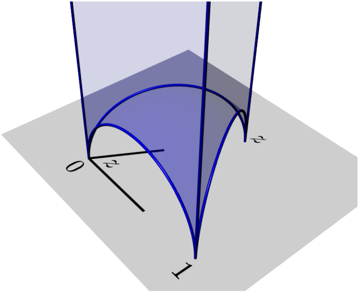
The shape of an ideal tetrahedron embedded in $\mathbb{H}^3$ is determined by a single complex "angle", associated to an edge.

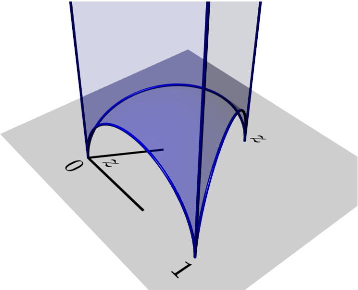


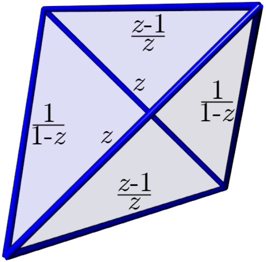
Geometric

1. In each tetrahedron, angles at opposite edges are the same.
2. If one angle is z, then the other two are $\frac{1}{1-z}$ and $\frac{z-1}{z}$.
3. Around each edge of $\mathcal{T}$, $\prod z = 1$.

For a geometric structure, all angles have $\mathrm{Im}(z) > 0$.

These conditions give us polynomials in the complex angles: Thurston's gluing equations.

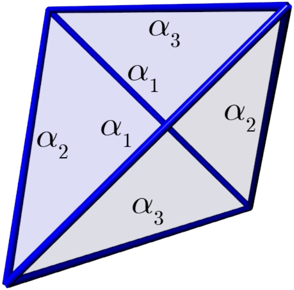




Angle structures

Associate angles (real numbers) to the edges of the tetrahedra of $\mathcal{T}$, so that:

1. In each tetrahedron, angles at opposite edges are the same.
2. In each tetrahedron, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
3. Around each edge of $\mathcal{T}$, $\sum \alpha = 2\pi$.

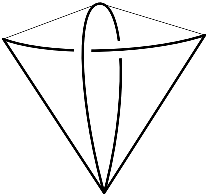


If all angles are in $(0, \pi)$ then this is a strict angle structure on $\mathcal{T}$.

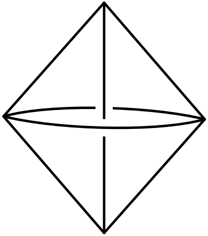
The angle structure equations can be thought of as a linearisation of the gluing equations.

Essential edges

A triangulation has essential edges if no edge can be homotoped into ∂M, keeping the endpoints on ∂M.

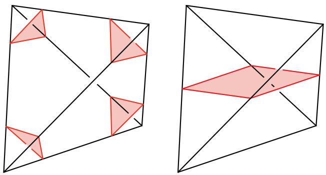


A triangulation has strongly essential edges if in addition, no edge can be homotoped to another edge, keeping the endpoints on ∂M.



Efficiency

A normal surface is made out of normal disks in each tetrahedron.



A triangulation is 0-efficient if there are no embedded normal S^2 or RP^2.

A triangulation is 1-efficient if in addition, there are no embedded normal Klein bottles, and the only embedded normal tori have no quadrilaterals.

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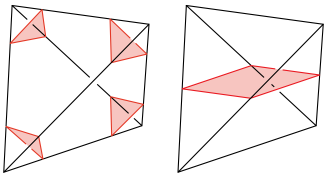

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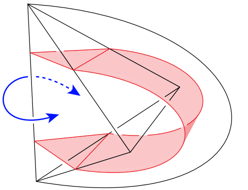
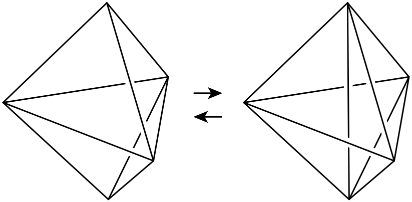
Example:

A triangulation with a degree 1 edge opposite a degree 2 edge is not 1-efficient.



The Pachner graph



Theorem (Matveev, Piergallini)

Any two ideal triangulations of M are connected by a sequence of 2-3 moves (except triangulations with a single tetrahedron).

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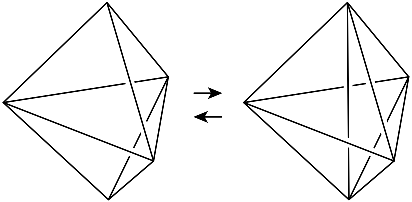

Theorem (Matveev, Piergallini)

Any two ideal triangulations of M are connected by a sequence of 2-3 moves (except triangulations with a single tetrahedron).

So $T(M)$ can be thought of as a connected graph, where two triangulations are connected by an edge if they are related by a 2-3 move.

Main question

Are the subgraphs of $T(M)$ corresponding to the various properties connected?

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Theorem (Hodgson, Hoffman, Dadd and Duan)

The geometric triangulations of the figure 8 knot complement are not connected.

There is however a unique ray of geometric triangulations, starting from the canonical two-tetrahedron triangulation.

Aside: how does SnapPy find geometric triangulations?

SnapPy performs heuristic moves, reducing the number of tetrahedra as much as possible. When it reaches a locally minimal triangulation, it tries to solve Thurston's gluing equations.

If this fails, it performs random 2-3 moves, then again simplifies and tries to solve the gluing equations.

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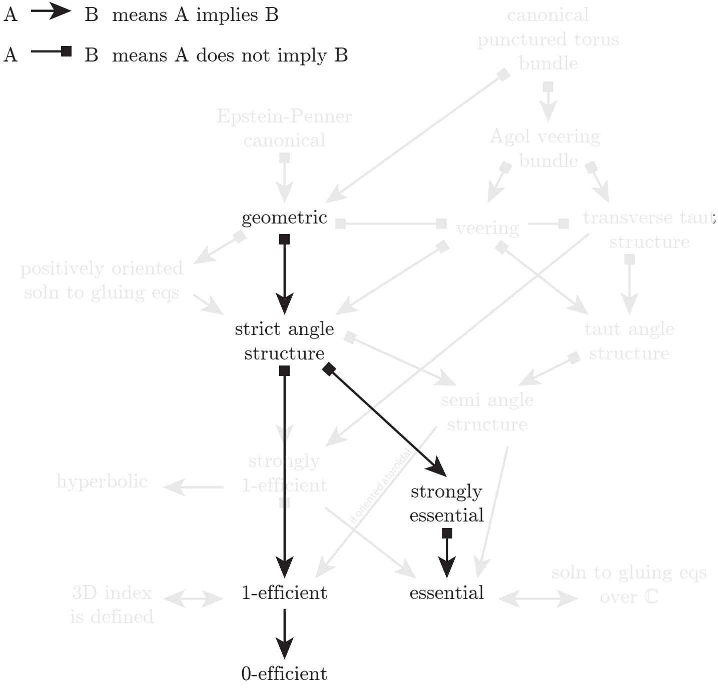
If this fails, it performs random 2-3 moves, then again simplifies and tries to solve the gluing equations.

This suggests that the geometric triangulations are easy to find, and that $T(M)$ is easy to navigate, but nobody knows why.

Showing connectivity

Let's start with an even weaker property: not having degree one edges.



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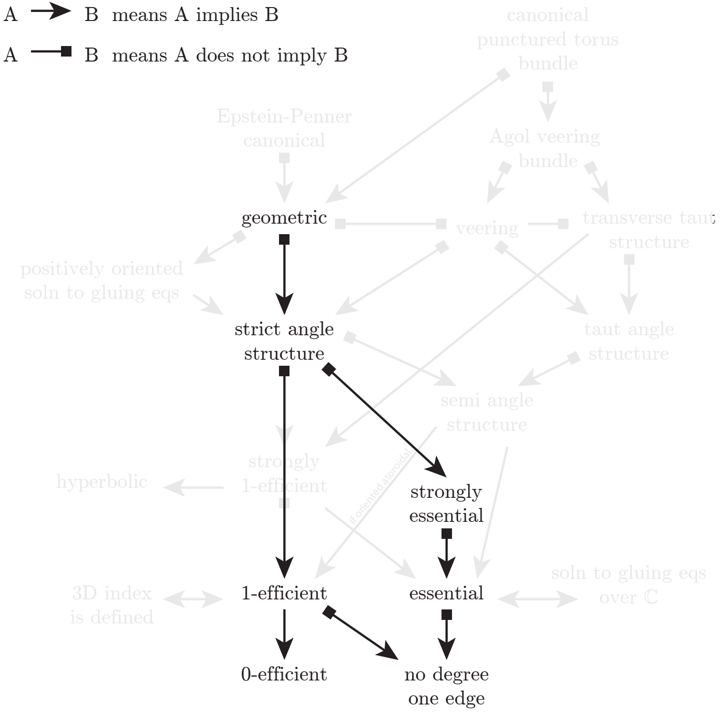

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Theorem (S)

The set of triangulations of a 3-manifold for which no edge is degree one is connected.

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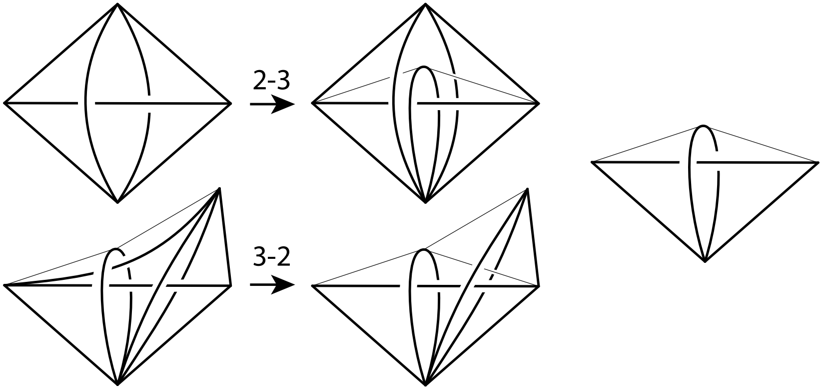
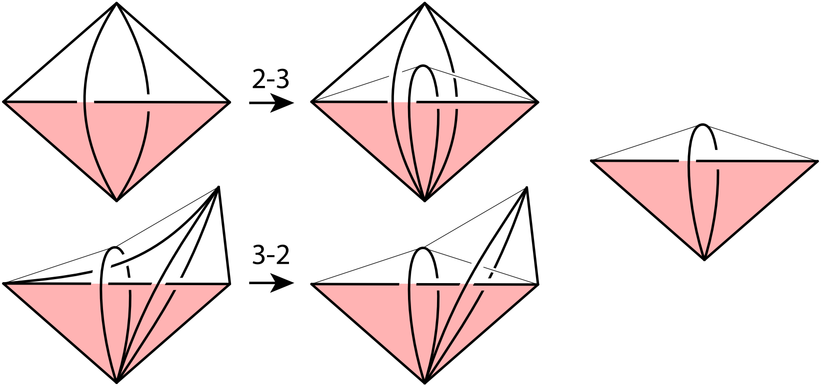
The set of triangulations of a 3-manifold for which no edge is degree one is connected.

Sketch proof:

By Matveev/Piergallini, there is some path between two triangulations of a manifold that do not have degree one edges. We have to sidestep around sections of the path that have degree one edges.



A 2-3 or 3-2 move can only produce a degree one edge from an edge that was previously degree two.



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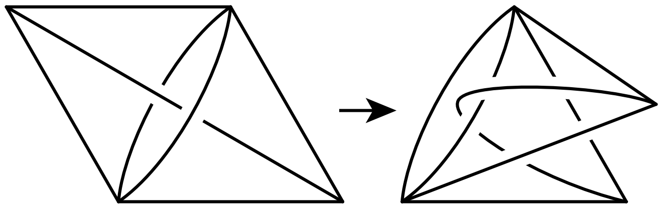
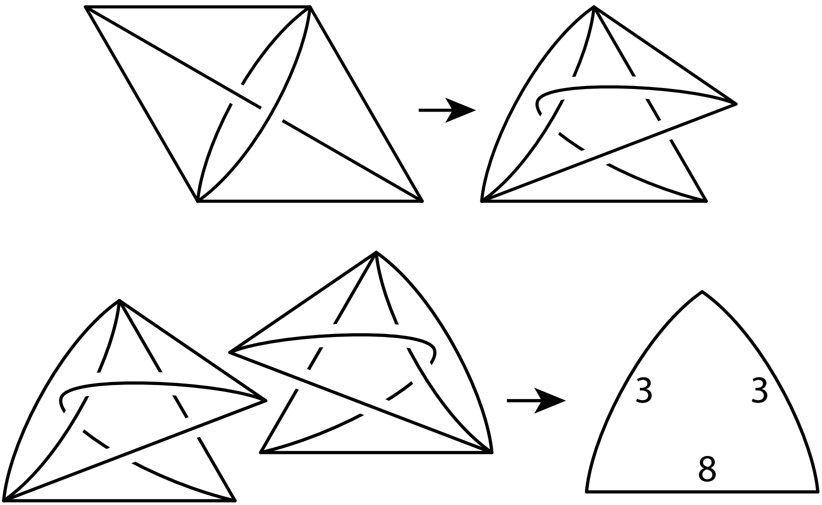
Just before the edge becomes degree one, the triangle that will be incident to the edge is already there, and remains there for the lifetime of the degree one edge.

The idea is to put something into the triangulation at that triangle that increases the degree of the edges incident to it.




3
3
8

The same kind of trick is unlikely to work for avoiding degree two edges.

A minimax approach might be more natural: choose a complexity measure such that minimal complexity paths should not involve "inefficient" features.

But an "efficient" path between two triangulations without degree two edges could go through a triangulation with a degree two edge.

So we may want to move to the more general setting of triangulations with strongly essential edges, or at least with no pairs of isotopic edges.

By using the bigon between two isotopic edges, we can find explicit sequences of moves to remove such pairs of edges.

Similar ideas may work to imitate crushing a normal surface via Pachner moves.

Thanks!