Ideal triangulations and components of the Character variety

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Deformation and Character Varieties

Two ways to think about deforming the structure of a 3-manifold M (through incomplete hyperbolic structures):

- Deformation variety of M, $\mathfrak{D}(M, \mathcal{T})$
  - Depends on a particular triangulation of M.
  - But to what extent?
- Character variety of M, $\mathcal{X}(M)$
  - Much harder to visualise.

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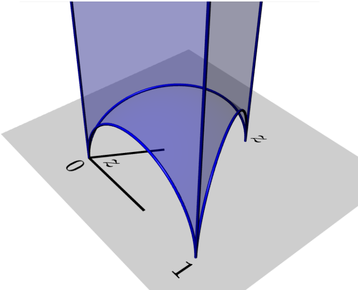

The shape of an ideal tetrahedron embedded in $\mathbb{H}^3$ is determined by a single complex dihedral angle.

Around an edge of the triangulation $\mathcal{T}$ of M the product of the complex dihedral angles must be 1.

These conditions give us polynomials in the complex angles, a solution to which will give us a (usually incomplete) hyperbolic structure. The set of such solutions is the deformation variety, $\mathfrak{D}(M; \mathcal{T})$.

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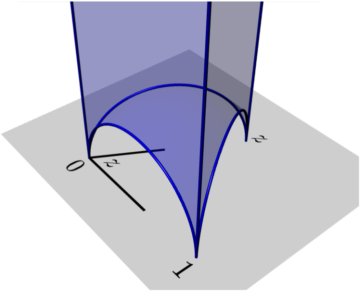

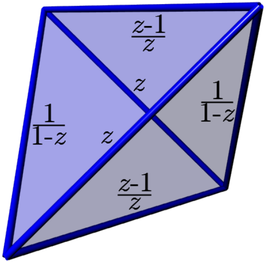

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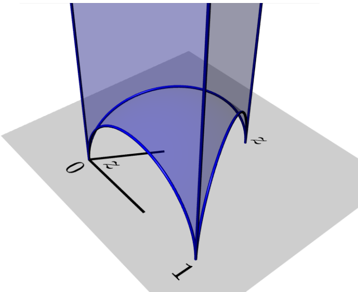

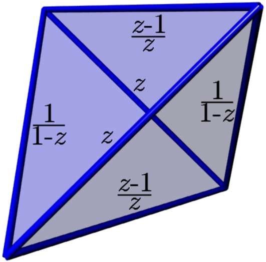

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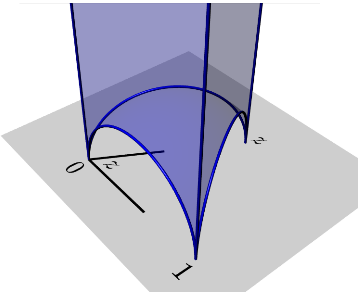

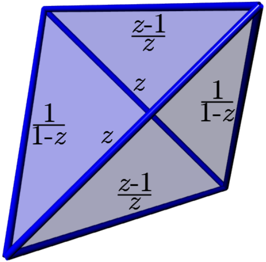

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The Character Variety

Look at the set of representations:

$$\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$$

To deal with "slack" from equivalent conjugate representations, consider characters:

$$\chi : \pi_1(M) \xrightarrow{\rho} \mathrm{PSL}_2(\mathbb{C}) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

It turns out that the set of characters also form a variety, $\mathcal{X}(M)$.

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What is the relationship between the varieties?

Theorem (Tillmann (Thurston etc.)) Let M^3 be a complete hyperbolic manifold of finite volume, with non-empty boundary consisting of a disjoint union of tori, and let $\mathcal{T}$ be an ideal (topological) triangulation of M. Then:

1. If $\mathfrak{D}(M;\mathcal{T})$ is non-empty, then there is an algebraic map $\chi_{\mathcal{T}} : \mathfrak{D}(M;\mathcal{T}) \to \mathcal{X}(M)$.
2. $\mathfrak{D}(M;\mathcal{T})$ is non-empty if and only if all edges of $\mathcal{T}$ are essential.
3. If $\mathfrak{D}(M;\mathcal{T})$ is non-empty, then $\mathcal{X}_0(M)$ is in the image of $\chi_{\mathcal{T}}$.

Here $\mathcal{X}_0(M)$ is the Dehn Surgery component of $\mathcal{X}(M)$, which contains the discrete and faithful character (corresponding to the complete structure). An edge is essential if it cannot be isotoped into ∂M.

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The upshot

In other words, if a triangulation of M "sees" a representation at all, then it will see the whole of the Dehn surgery component.

What about the other components (if there are any)?

Assuming we restrict our attention to sensible triangulations (i.e. that have only essential edges), does which triangulation we choose matter as to which other components we can "see"?

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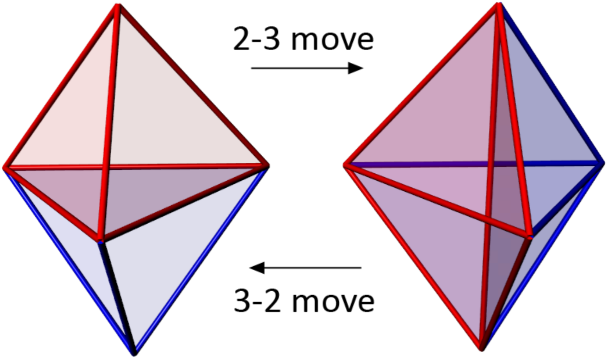
Question: Is there some way we can deal with this unpleasant dependence on the triangulation?

The 2-3 and 3-2 moves



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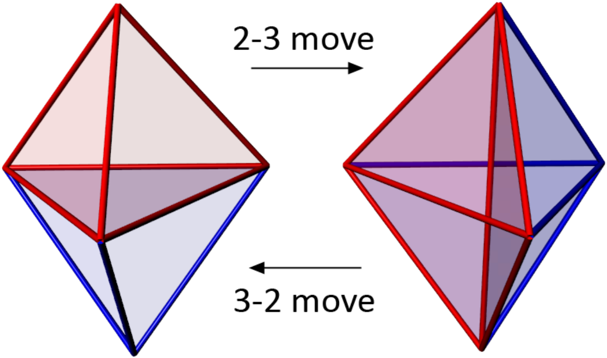
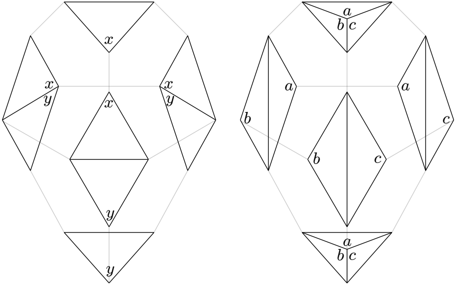


Theorem (Matveev) Any two ideal triangulations of a given manifold M are connected by a sequence of 2-3 and 3-2 moves.

What do these moves do to the deformation variety?

How are $\mathfrak{D}(M;\mathcal{T}_2)$ and $\mathfrak{D}(M;\mathcal{T}_3)$ related?
Outside of the six sided polyhedron, nothing changes...



$$a = \frac{y-1}{y(1-x)}$$
$$b = \frac{x-1}{x(1-y)}$$
$$c = \frac{1}{ab}$$

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Everything translates unless $x = 1/y$, which corresponds to the top and bottom vertices of the two tetrahedra actually being in the same place on S^2_∞. The corresponding three tetrahedra are all degenerate.

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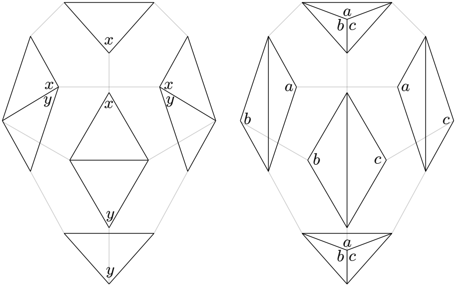

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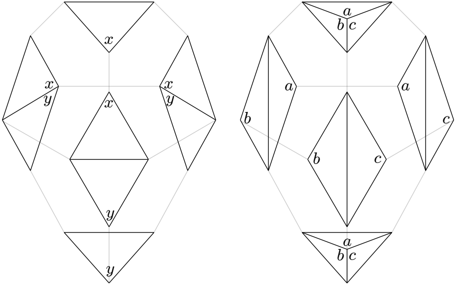

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An ideal point of $\mathcal{X}(M)$ is a limit of characters in $\mathcal{X}(M)$ such that for some $\gamma \in \pi_1(M)$, the characters evaluated on γ diverge to ∞. "The length of some path becomes infinite".

An ideal point of $\mathfrak{D}(M;\mathcal{T})$ is a limit of complex angles of the tetrahedra such that at least one angle converges to 0, 1 or ∞.

Some ideal points of $\mathfrak{D}(M;\mathcal{T})$ correspond to ideal points of $\mathcal{X}(M)$, but not all. The ideal point we can get after a 2-3 move is an example of such a "fake" ideal point.

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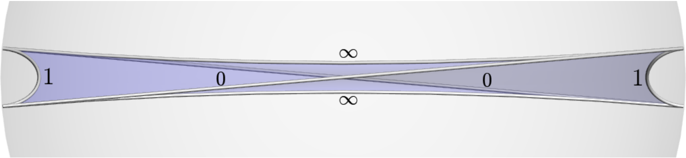

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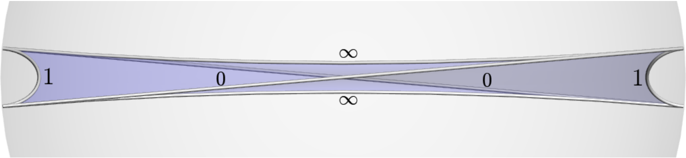

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How could different triangulations "see" different components of $\mathcal{X}(M)$?

Isolated "fake" ideal points are not a big deal: we should expect equivalence of our deformation varieties only up to birational isomorphisms.

But what if we have the "$x = 1/y$" relation for an entire component?

Then $\mathfrak{D}(M; \mathcal{T}_2)$ would be able to see the component, but $\mathfrak{D}(M; \mathcal{T}_3)$ would not (complex dihedral angles of 0, 1 or ∞ aren't allowed in the deformation variety).

Does this actually happen?

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IGKT knot (8_{18})

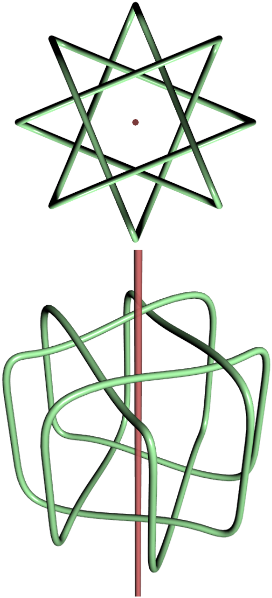
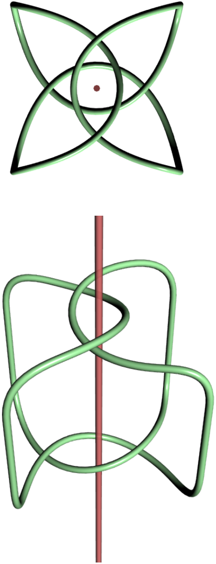


Figure 8 knot (4_1)



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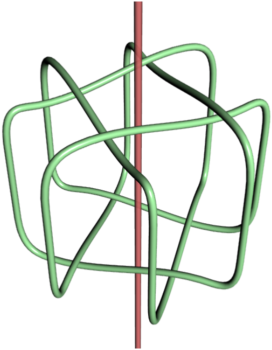

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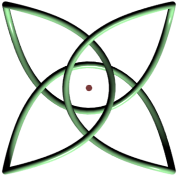
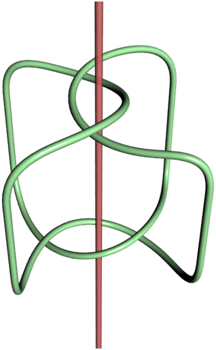




$\mathcal{X}(8_{18})$ has 2 components

The 2-fold covering map $8_{18} \to$ fig 8 induces:

$$\Psi : \pi_1(S^3 \setminus 8_{18}) \to \pi_1(S^3 \setminus \text{fig } 8)$$

We get (at least) two components of $\mathcal{X}(S^3 \setminus 8_{18})$:

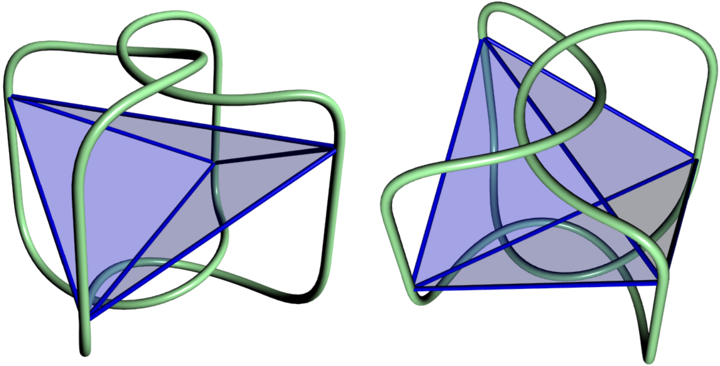
- $\mathcal{X}_0$: The Dehn surgery component of $\mathcal{X}(S^3 \setminus 8_{18})$
- $\mathcal{X}_1$: The component containing characters that correspond to representations into $\mathrm{PSL}_2(\mathbb{C})$ which factor through Ψ

The complete structure for $S^3 \setminus 8_{18}$ ($\rho_{8_{18}}$) is in $\mathcal{X}_0$.
"Twice" the complete structure for $S^3 \setminus$ fig 8 ($\rho_{\text{fig } 8}$) is in $\mathcal{X}_1$.

Thanks to Eric Chesebro for telling me about this example.

Triangulation of the Figure 8 knot complement

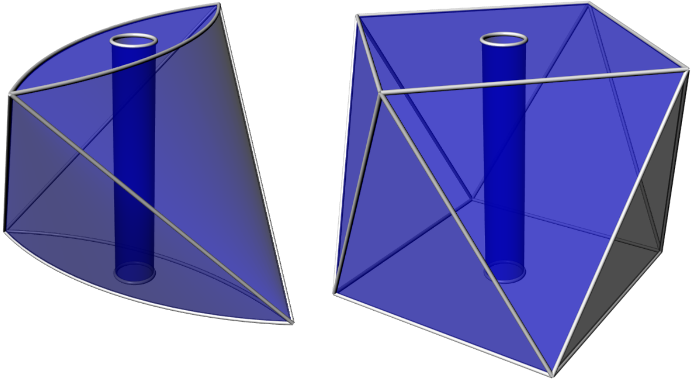
As in Thurston, we can cut up the complement of the Figure 8 knot into two tetrahedra.



(Shown are 2 views of the same thing.) There is another tetrahedron on the "outside" of the knot, the top two faces of which glue to the top two faces of the inside tetrahedron, and similarly for the bottom faces.

"Two fold cover of the triangulation"

The braid axis goes through both the inside and outside tetrahedra from top to bottom, through the top and bottom edges. To see what happens when we take the 2 fold cover, first push apart the top edge to form a bigon, then drill out the braid axis.

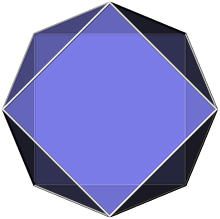


The resulting 2-fold cover polyhedron is a 4-sided antiprism.

Triangulate the antiprisms

So the exterior of 8_{18} is made up of two antiprisms. We need to triangulate these:

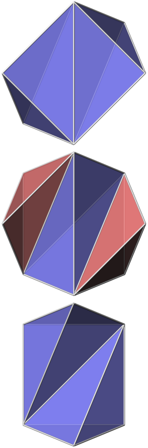
Antiprism:



Layered:

We subdivide the antiprism into 7 tetrahedra, shown in 3 layers. Note that all but one of the tetrahedra is paired with another under rotation by π along the braid axis. The other antiprism can be subdivided in the same way, making sure that the way that the squares are subdivided is consistent.

Denote this triangulation of $M = S^3 \setminus 8_{18}$ by $\mathcal{T}_A$.



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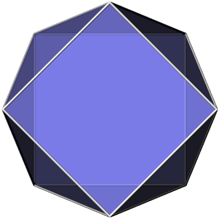

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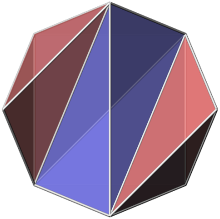
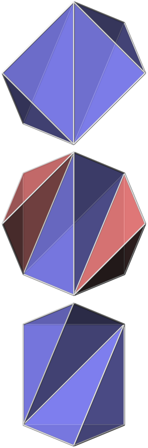


We subdivide the antiprism into 7 tetrahedra, shown in 3 layers. Note that all but one of the tetrahedra is paired with another under rotation by π along the braid axis. The other antiprism can be subdivided in the same way, making sure that the way that the squares are subdivided is consistent.

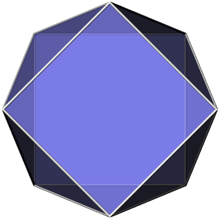
Denote this triangulation of $M = S^3 \setminus 8_{18}$ by $\mathcal{T}_A$.



Triangulate the antiprisms

So the exterior of 8_{18} is made up of two antiprisms. We need to triangulate these:

Antiprism:



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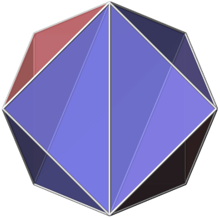

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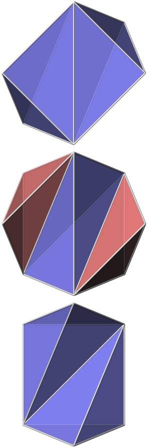

Claim: $\mathfrak{D}(M; \mathcal{T}_A)$ cannot produce any representation $\rho_{\text{fig } 8} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ which factors through the two fold cover Ψ.

The problem is the edges of the triangulation that are "diameters", that is whose endpoints map to the same place under the 2-fold covering map.

Let $\widetilde{M}$ be the universal cover of M with the induced triangulation. The idea is to assume that some $x \in \mathfrak{D}(M; \mathcal{T}_A)$ maps to some $\rho_{\text{fig } 8}$, and see where the ideal vertices of tetrahedra in $\widetilde{M}$ would have to go in $S^2_\infty = \partial\mathbb{H}^3$.

It turns out that ends of a diameter must get mapped to the same point of S^2_∞, and so any tetrahedron with a diameter as an edge would have to be degenerate, and so x cannot exist.

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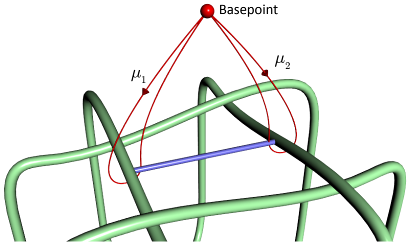
Ends of diameters go to the same point of S^2_∞

The vertices of ideal tetrahedra in $\widetilde{M}$ correspond to $\mathbb{Z} \times \mathbb{Z}$ subgroups in $\pi_1(M)$. Any such subgroup can be conjugated to any other by some $w \in \pi_1 M$. In general, if $x \in \mathfrak{D}(M; \mathcal{T})$ maps to ρ, then two ideal vertices in $\widetilde{M}$ conjugate by w correspond to two points of S^2_∞ such that $\rho(w)$ takes one to the other.



Longitudes λ_1, λ_2 are not shown.

$< \mu_1, \lambda_1 >$ is conjugate to, but not equal to $< \mu_2, \lambda_2 >$.

Here w is trivial after the map Ψ, so these two vertices would have to be in the same place in S^2_∞, so x cannot exist.

Retriangulating?

So $\mathfrak{D}(M; \mathcal{T}_A)$ does not contain in its image any $\rho_{\text{fig } 8}$.

The component $\mathcal{X}_1$ is 1 dimensional (this is not obvious), so it consists only of characters of representations that factor through *Psi*, so we can conclude that $\mathfrak{D}(M; \mathcal{T}_A)$ does not see $\mathcal{X}_1$.

In general, we expect (?) that a given triangulation should see all but slices of a higher dimensional component.

What if we retriangulate so that we don't have any diameters?

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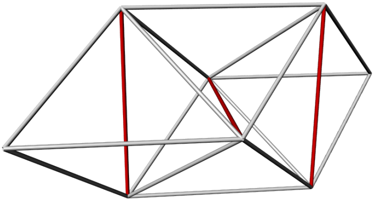

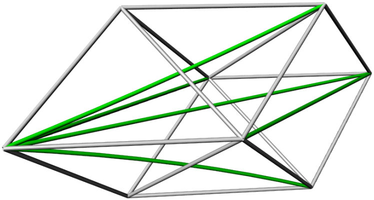

After a number of 2-3 and 3-2 moves, we obtain a new triangulation, $\mathcal{T}_B$, which has 20 tetrahedra, and which does not contain any diameters as edges. Does $\mathfrak{D}(M; \mathcal{T}_B)$ see $\mathcal{X}_1$?

Yes. We can see this "directly" by throwing all 20 gluing equations into Mathematica and doing some calculations:

- Set the "paired" tetrahedra to have the same complex angles.
- Set "$x = 1/y$" where appropriate to collapse the diameters.
- Solve for values of some angles in terms of others.

In the end most gluing equations are identically equal to 1 and there are four others, which are precisely 2 copies of the 2 gluing equations for the figure 8 knot!

What is the relationship between the varieties?

Recall:

Theorem (Tillmann (Thurston etc.)) Let M^3 be a complete hyperbolic manifold of finite volume, with non-empty boundary consisting of a disjoint union of tori, and let $\mathcal{T}$ be an ideal (topological) triangulation of M. Then:

1. If $\mathfrak{D}(M;\mathcal{T})$ is non-empty, then there is an algebraic map $\chi_{\mathcal{T}} : \mathfrak{D}(M;\mathcal{T}) \to \mathcal{X}(M)$.
2. $\mathfrak{D}(M;\mathcal{T})$ is non-empty if and only if all edges of $\mathcal{T}$ are essential.
3. If $\mathfrak{D}(M;\mathcal{T})$ is non-empty, then $\mathcal{X}_0(M)$ is in the image of $\chi_{\mathcal{T}}$.

Here $\mathcal{X}_0(M)$ is the Dehn Surgery component of $\mathcal{X}(M)$, which contains the discrete and faithful character (corresponding to the complete structure). An edge is essential if it cannot be isotoped into ∂M.

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3. If $\mathfrak{D}(M;\mathcal{T})$ is non-empty, then $\mathcal{X}_0(M)$ is in the image of $\chi_{\mathcal{T}}$. So $\mathfrak{D}(M;\mathcal{T}_B)$ also sees $\mathcal{X}_0$.

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Tying up a loose end

Since all edges of $\mathcal{T}_B$ are essential, and the only edges of $\mathcal{T}_A$ that are not in $\mathcal{T}_B$ are the two diameters, we need only check that these edges are essential to show that $\mathfrak{D}(M;\mathcal{T}_A)$ is non-empty and does in fact see $\mathcal{X}_0$.

In this case this is not too hard, knowing something about the essential surfaces in M.

In conclusion, the triangulation does matter as to which components of the character variety are mapped to by the respective deformation varieties.

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...which is annoying.

Independence of triangulation?

We would prefer to be able to work with any given triangulation of a manifold, and not worry that we may miss some components of $\mathcal{X}(M)$. In particular, if we don't know how some other component comes about, we wouldn't know which edges are presumably non-essential, and would have no way to know how to try to retriangulate.

So we would like to modify our variety in some way, keeping the same triangulation the same, but somehow being able to deal with degenerate tetrahedra.

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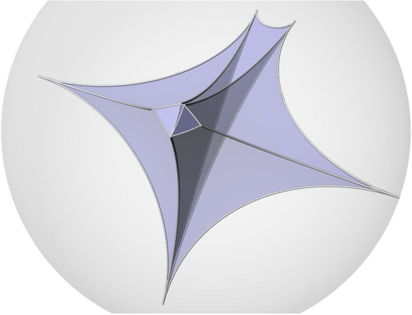
Caveat: I don't know why this works

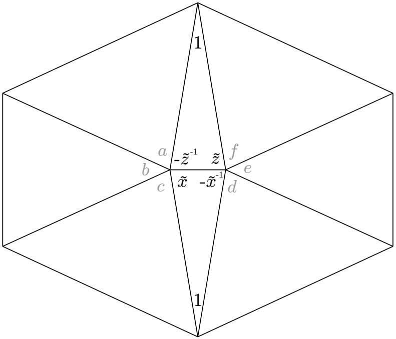
- For x an angle that is supposed to be 0, set $x = \zeta\tilde{x}$.
- Angles that were supposed to be ∞ now look like $\frac{\zeta\tilde{x}-1}{\zeta\tilde{x}}$
- Angles that were supposed to be 1 now look like $\frac{1}{1-\zeta\tilde{x}}$
- Cancel factors of ζ in the gluing equations and set $\zeta = 0$ (the ζ from 0 angles cancel with the $\frac{1}{\zeta}$ from ∞ angles)
- We get a new variety, $\widetilde{\mathfrak{D}}(M;\mathcal{T})$ (which depends on our choice of degeneration)

Do this for $\mathfrak{D}(M;\mathcal{T}_A)$ to collapse the diameters to produce $\widetilde{\mathfrak{D}}(M;\mathcal{T}_A)$. Solve for variables, and we again end up with most gluing equations identically equal to 1 and the others being 2 copies of the 2 gluing equations for the figure 8 knot!

(there is a very small lie here)

But there is no limit here, these tetrahedra are degenerate throughout the entire component, so I don't know why this works.

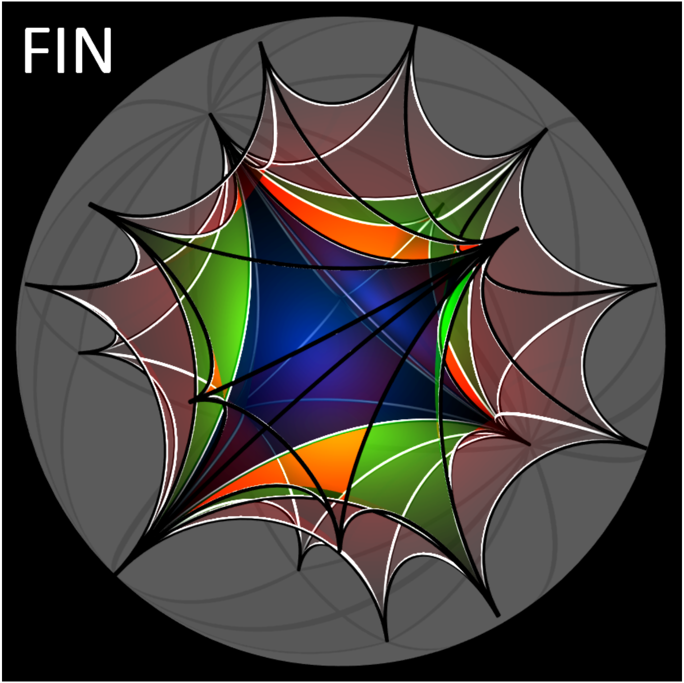




These tilde variables somehow have the effect of transferring information from one edge to the other, effectively combining the gluing equations of the two edges into one:

$$abc = \frac{1}{-\tilde{z}^{-1}\tilde{x}}$$
$$def = \frac{1}{-\tilde{x}^{-1}\tilde{z}}$$
$$(-\tilde{z}^{-1}\tilde{x}) = \frac{1}{-\tilde{x}^{-1}\tilde{z}}$$
$$\Rightarrow abcdef = 1$$

Something more complicated happens at the other ends of the merged edges.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