

# SMOOTH COMPONENTS OF SPRINGER FIBERS

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ABSTRACT. This article studies components of Springer fibers for  $\mathfrak{gl}(n)$  that are associated to closed orbits of  $GL(p) \times GL(q)$  on the flag variety of  $GL(n)$ ,  $n = p + q$ . These components occur in any Springer fiber. We show that, in contrast to the case for arbitrary components, these components are smooth varieties, and are invariant under a maximal torus of  $GL(n)$ . This is done by using results of Barchini and Zierau ([3]) to give a precise description of these components as iterated bundles. This description implies that such a component is a fiber bundle over a generalized flag variety for a smaller general linear group, with fibers isomorphic to a component of the Springer fiber for a (different) smaller general linear group. We prove that if  $\mathcal{L}$  is a line bundle on the flag variety associated to a dominant weight, then the higher cohomology groups of the restriction of  $\mathcal{L}$  to these components vanish. We derive some consequences of localization theorems in equivariant cohomology and  $K$ -theory, applied to these components. In the appendix we identify the tableaux corresponding to these components, under the bijective correspondence between components of Springer fibers for  $GL(n)$  and standard tableaux.

## INTRODUCTION

Let  $G$  be a complex reductive algebraic group with flag variety  $\mathfrak{B}$ . The moment map  $\mu : T^*\mathfrak{B} \rightarrow \mathfrak{g}^*$  is called the Springer resolution. If we identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by using a nondegenerate  $G$ -invariant bilinear form, then the image of  $\mu$  is the nilpotent cone in  $\mathfrak{g}$ , and the Springer fibers are the inverse images of nilpotent elements of  $\mathfrak{g}$ . A Springer fiber  $\mu^{-1}(X)$  can be identified with its image under the projection  $T^*\mathfrak{B} \rightarrow \mathfrak{B}$ . This image is the set of Borel subalgebras of  $\mathfrak{g}$  that contain  $X$ , or equivalently, the set  $\mathfrak{B}^u$  of points in  $\mathfrak{B}$  fixed by the unipotent element  $u = \exp X$ .

In the classic papers [26] and [27], Springer constructed a representation of the Weyl group  $W$  on the top degree homology of a Springer fiber. Subsequently, the Springer fibers and these Springer representations have played an important role in several areas in representation theory. Nevertheless, the geometry of the Springer fibers is still not well-understood. The fixed point scheme  $\mathfrak{B}^s$  of a semisimple element  $s \in G$  is smooth ([20]) and stable under the action of a maximal torus of  $G$ . In contrast, the Springer fibers  $\mathfrak{B}^u$  are almost always singular. Indeed, they must be singular because the different irreducible components of  $\mathfrak{B}^u$  often intersect. The individual components

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of  $\mathfrak{B}^u$  can be singular as well. Moreover, the components are not in general stable under the action of a maximal torus of  $G$ . These facts all complicate the study of the Springer fibers.

In this paper, building on the work of Barchini and Zierau ([3]), we study certain components of Springer fibers in case  $G = GL(n)$ . As above, we view the Springer fiber as a subscheme of the flag variety  $\mathfrak{B}$ . Our main result is that these components are isomorphic to iterated fiber bundles constructed using subgroups of  $G$ . Thus, in contrast to the general case, these components are smooth and stable under the action of a maximal torus  $H$  of  $G$ . Using this description of the components, we calculate Betti numbers, we obtain a character formula related to associated cycles of discrete series representations, and we express the (equivariant) cohomology and  $K$ -theory classes defined by the components in terms of Schubert bases.

We now describe our results in more detail. Let  $(G, K)$  denote the pair of groups  $(G, K) = (GL(n), GL(p) \times GL(q))$ ,  $p + q = n$ . The group  $K$  acts with finitely many orbits on  $\mathfrak{B}$ . Fix a closed  $K$ -orbit  $\mathcal{Q}$ . Note that  $\mathcal{Q}$  is isomorphic to the flag variety for  $K$ . Let  $\gamma : T_{\mathcal{Q}}^* \mathfrak{B} \rightarrow \mathfrak{g}$  denote the restriction of  $\mu$  to the conormal bundle  $T_{\mathcal{Q}}^* \mathfrak{B}$ . The image  $\gamma(T_{\mathcal{Q}}^* \mathfrak{B})$  is the closure of a single  $K$ -orbit  $K \cdot f$ ;  $f$  is called generic. The inverse image  $\gamma^{-1}(f)$  is a single component of the Springer fiber  $\mu^{-1}(f)$ . We say that such a component is associated to the closed  $K$ -orbit  $\mathcal{Q}$ . We view the component  $\gamma_{\mathcal{Q}}^{-1}(f)$  as a subvariety of  $\mathfrak{B}$ ; in fact,  $\gamma_{\mathcal{Q}}^{-1}(f) \subset \mathcal{Q} \subset \mathfrak{B}$ .

In [3], the authors define a sequence of pairs  $(G_0, K_0) = (G, K) \supset (G_1, K_1) \supset \cdots \supset (G_m, K_m)$ , where  $(G_i, K_i)$  is a pair of the same type as  $(G, K)$ . They define elements  $f_i \in \mathfrak{g}_i = \text{Lie } G_i$  so that  $f = \sum f_i$  is generic. They also define parabolic subgroups  $Q_i = L_i U_i \subset H K_i$ , where  $H$  is the diagonal torus. The main results of [3] (see Proposition 1.8 below) imply that

$$\gamma_{\mathcal{Q}}^{-1}(f) = Q_m \cdots Q_1 Q_0 \cdot \mathfrak{b} = L_m \cdots L_1 L_0 \cdot \mathfrak{b} \subset G/B, \quad (1)$$

where  $B$  is a Borel subgroup of  $G$ , chosen so that the orbit  $\mathcal{Q}$  is  $K \cdot \mathfrak{b}$ . Let  $R_i = Q_i \cap Q_{i-1}$  and define

$$X_m = Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \cdots \times_{R_2} Q_1 \times_{R_1} Q_0 / R_0;$$

$X_m$  is a bundle over  $Q_m/R_m$  with fibers isomorphic to  $X_{m-1}$ . Equation (1) implies that the map  $F : X_m \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  defined by  $F([q_m, q_{m-1}, \dots, q_0]) = q_m \cdots q_1 q_0 \cdot \mathfrak{b}$  is surjective. The main result of this paper (Theorem 2.20) is that  $F$  is an isomorphism of algebraic varieties. This theorem implies that the component  $\gamma_{\mathcal{Q}}^{-1}(f)$  is a fiber bundle over  $Q_m/R_m$ , which is a generalized flag variety for a product of general linear groups, with fiber isomorphic to a component of the same type for a smaller pair  $(G', K')$  (Corollary 2.35).

The description of  $\gamma_{\mathcal{Q}}^{-1}(f)$  as an iterated bundle has a number of applications. It implies that  $\gamma_{\mathcal{Q}}^{-1}(f)$  is smooth and  $H$ -invariant, and makes it easy to calculate the Betti

numbers of  $\gamma_{\mathcal{Q}}^{-1}(f)$  (see Remark 2.38). Using our description of  $\gamma_{\mathcal{Q}}^{-1}(f)$  we determine the  $H$ -fixed points (Proposition 4.1), and also the weights of  $H$  acting on the tangent spaces at these points (Corollary 4.4). This makes it possible to apply localization theorems in equivariant  $K$ -theory and Borel-Moore homology. Using these theorems we obtain a formula for the character of  $H$  acting on  $\sum_i (-1)^i H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$ , where  $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$  is an invertible sheaf on  $\gamma_{\mathcal{Q}}^{-1}(f)$  corresponding to the weight  $\tau$  of  $H$  (Theorem 4.6). This formula is of interest because of a result of J.-T. Chang [6], which states that the dimension of  $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$  is the multiplicity of  $\overline{K} \cdot f$  in the associated cycle of a discrete series representation of  $U(p, q)$ . If  $\tau$  is sufficiently dominant, the higher cohomology groups vanish, so our formula gives the character of  $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$ . As another application of our results and localization theorems, we obtain formulas expressing the classes defined by  $\gamma_{\mathcal{Q}}^{-1}(f)$  in equivariant cohomology and  $K$ -theory in terms of Schubert bases (Theorems 4.8 and 4.9). These formulas imply corresponding non-equivariant formulas, answering (for these components) a question raised by T. A. Springer ([28]), and answered in some special cases by J. Güemes ([18]).

Typically one wants to understand the components in some Springer fiber  $\mu^{-1}(f)$ . In the approach taken here (and in [3]) the ‘ $f$ ’ is a moving target. However, it is readily seen that often two closed orbits  $\mathcal{Q}$  and  $\mathcal{Q}'$  give generic elements  $f$  and  $f'$  in the *same* nilpotent  $G$ -orbit. When this occurs, by translating by some  $g \in G$ , the components  $\gamma_{\mathcal{Q}}^{-1}(f)$  and  $\gamma_{\mathcal{Q}'}^{-1}(f')$  may be identified with components in a single Springer fiber. The purpose of the appendix is to label the components studied in this article in terms of the usual parametrization of components. This will therefore identify the components of any Springer fiber that are of the form  $\gamma_{\mathcal{Q}}^{-1}(f)$  (under the above identification) for some closed  $\mathcal{Q}$  and generic  $f$ . Recall that the nilpotent orbits in  $\mathfrak{gl}(n)$  are parametrized by partitions of  $n$ . If  $f$  is in the orbit corresponding to the partition  $\lambda$ , the components of the Springer fiber  $\mu^{-1}(f)$  are parametrized by the set of standard tableaux on the shape of  $\lambda$ ; see [25] and [29]. In the appendix, we describe the standard tableaux corresponding to the components we consider in this paper. In fact, the Springer fiber corresponding to *any* nilpotent orbit contains components of the type considered in this article.

Results about the smoothness of components of Springer fibers appear in the literature. An example of a nonsmooth component of a Springer fiber in  $\mathfrak{sl}(6)$  is given in [31]. More recently L. Fresse ([10]) has determined exactly which components are smooth for Springer fibers of nilpotents in  $\mathfrak{gl}(n)$  having tableau with exactly two columns. Springer fibers of nilpotent elements in  $\mathfrak{gl}(n)$  have been studied by F. Fung ([12]) in the case where the Young diagram of  $f$  is either of hook shape or has two rows. He shows that the components are iterated bundles and he computes the Betti numbers of components in these cases. Combinatorial formulas for Betti numbers of Springer fibers are contained in the work of G. Lusztig (see for example [23]). A direct computation

for  $\mathfrak{gl}(n)$  is given in [9]. Note that our formulas are for Betti numbers of individual components. Some of our applications of our main result use a description of  $H$ -fixed points (Prop. 4.1); fixed points are determined in some special cases in [11]. The components of Springer fibers associated to closed  $K$ -orbits in  $\mathfrak{B}$  have been studied for other classical groups by Barchini and Zierau, and they have obtained descriptions that are similar to that in the case of  $(G, K) = (GL(n), GL(p) \times GL(q))$ ,  $p + q = n$ . We believe that these descriptions imply that in the other classical cases these components are again isomorphic to iterated bundles. This will be pursued elsewhere.

**Notation.** We work over the field of complex numbers. We fix once and for all the pair of complex groups

$$(G, K) = (GL(n), GL(p) \times GL(q)), \quad p + q = n.$$

Then  $K$  is the fixed point group of the involution  $\Theta$  given by conjugation by the matrix  $I_{p,q} := \text{diag}(I_p, I_q)$ . The decomposition of  $\mathfrak{g}$  into  $\pm 1$ -eigenspaces is written as  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ .

The diagonal Cartan subgroup is denoted by  $H$ . As is customary we denote the Lie algebras of  $G, K, H$ , etc. by  $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ , etc. The root system  $\Delta(\mathfrak{h}, \mathfrak{g})$  is  $\{\epsilon_i - \epsilon_j : i \neq j\}$ , where  $\epsilon_k \in \mathfrak{h}^*$  is defined by  $\epsilon_k(\text{diag}(z_1, \dots, z_n)) = z_k$ . We will consider many positive systems for  $\Delta(\mathfrak{h}, \mathfrak{g})$ . However, we fix once and for all the following system of positive roots of  $\mathfrak{h}$  in  $\mathfrak{k}$ :

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq p \text{ or } p + 1 \leq i < j \leq n\}.$$

For each root  $\epsilon_i - \epsilon_j$ , we let  $X_{i,j}$  be the root vector equal to the matrix with a one in the  $(i, j)$  entry and zeros elsewhere. If  $L$  is a reductive subgroup of  $K$  containing the Cartan subgroup  $H$ , then the Weyl group is denoted by  $W(L)$ .

## 1. PRELIMINARIES

A detailed description of the components of the Springer fibers associated to closed  $K$ -orbits in  $\mathfrak{B}$  is given in [3]. Since this description plays a key role in the results of this article, we begin by carefully describing certain statements in [3]. Then we will give some consequences of these statements that will be needed later in the article.

**1.1. Components associated to closed  $K$ -orbits.** The closed  $K$ -orbits in  $\mathfrak{B}$  are in one-to-one correspondence with positive systems  $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$  that contain  $\Delta_c^+$ . Such a one-to-one correspondence is given by associating to  $\Delta^+$  the Borel subalgebra  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ ,  $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(-\alpha)}$ . Then  $\mathcal{Q} = K \cdot \mathfrak{b}$  is the corresponding closed  $K$ -orbit in  $\mathfrak{B}$ .

Let us fix a positive system  $\Delta^+$  containing  $\Delta_c^+$ , thus fixing a corresponding closed  $K$ -orbit  $\mathcal{Q} = K \cdot \mathfrak{b}$ . As described in [3, Section 2], the restriction of the moment map  $\mu : T^*\mathfrak{B} \rightarrow \mathfrak{g}^*$  to the conormal bundle  $T_{\mathcal{Q}}^*\mathfrak{B}$  may be identified with a map  $\gamma_{\mathcal{Q}} :$

$K \times_{B \cap K} (\mathfrak{n}^- \cap \mathfrak{p}) \rightarrow \mathcal{N}_\theta$ , where  $\mathcal{N}_\theta$  is the variety of nilpotent elements of  $\mathfrak{g}$  contained in  $\mathfrak{p}$ . The map  $\gamma_Q$  is given by the formula

$$\gamma_Q(k, Y) = k \cdot Y \quad (:= \text{Ad}(k)(Y)).$$

It is a well-known fact that the image of  $\gamma_Q$  is the closure of a single  $K$ -orbit in  $\mathcal{N}_\theta$ . An element  $f$  of  $\mathfrak{n}^- \cap \mathfrak{p}$  is said to be *generic in*  $\mathfrak{n}^- \cap \mathfrak{p}$  when  $\gamma_Q(T_Q^* \mathfrak{B}) = \overline{K \cdot f}$ .

Our first goal is to describe a particular generic element  $f$  in  $\mathfrak{n}^- \cap \mathfrak{p}$ . Note that each positive system  $\Delta^+$  containing  $\Delta_c^+$  is defined by a  $\Delta$ -regular,  $\Delta_c^+$ -dominant  $\lambda \in \mathfrak{h}^*$  by  $\Delta^+ = \{\alpha : \langle \alpha, \lambda \rangle > 0\}$ . The notation  $\lambda = (\lambda_1, \dots, \lambda_n)$  is used for  $\lambda = \sum \lambda_i \epsilon_i$ . We build  $f$  from  $\lambda$  inductively; first  $f_0$  is specified (as a sum of certain root vectors) and a subgroup  $G_1$  (a lower rank general linear group) of  $G$  is determined. Then  $f_1$  is chosen in  $\mathfrak{g}_1$  and a subgroup  $G_2$  of  $G_1$  is given by the same procedure. One continues, obtaining  $f_0, f_1, \dots, f_{m-1}$ ; then  $f = f_0 + \dots + f_{m-1}$  is our generic element. The inductive procedure is easily described by forming an ‘array’ from  $\lambda$ . This array consists of two rows of numbered dots and is constructed as follows. If the greatest coordinate of  $\lambda$  is among the first  $p$  coordinates then place the first dot in the upper row, otherwise place it in the lower row. Working from left to right, place the next dot in the upper row if the next greatest coordinate of  $\lambda$  is among the first  $p$  coordinates and in the lower row otherwise. Continue in this manner. The  $j^{\text{th}}$  dot (counting from the left) is in the upper row exactly when the  $j^{\text{th}}$  greatest coordinate of  $\lambda$  is among the first  $p$  coordinates. Now number the dots in the upper row with  $1, 2, \dots, p$  and those in the lower row by  $p+1, p+2, \dots, p+q = n$ . For example, if  $(G, K)$  is the pair  $(GL(7), GL(4) \times GL(3))$  and  $\lambda = (7, 6, 4, 3, 5, 2, 1)$ , then the array is

$$\begin{array}{ccccccc} 1 & 2 & & 3 & 4 & & \\ \bullet & \bullet & & \bullet & \bullet & & \\ & & & & & & \\ & & & 5 & & 6 & 7 \\ & & & \bullet & & \bullet & \bullet \end{array}$$

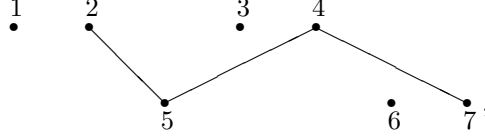
We define a *block* in the array to be a set of dots in the array that is maximal with respect to the properties (i) all dots lie in the same row and (ii) the dots are consecutive. In the example the blocks are  $\{1, 2\}, \{5\}, \{3, 4\}$  and  $\{6, 7\}$ .

Suppose that our array has  $N$  blocks. Define  $j_1, \dots, j_N \in \{1, 2, \dots, n\}$  so that  $j_i$  is the label of the dot farthest to the right in the  $i^{\text{th}}$  block. Set

$$f_0 = \sum_{i=1}^{N-1} X_{j_{i+1}, j_i}.$$

Note that  $f_0 \in \mathfrak{n}^- \cap \mathfrak{p}$ . (Each  $X_{j_{i+1}, j_i}$  is noncompact since blocks alternate between lying in the upper and lower rows. Each root  $\epsilon_{j_{i+1}} - \epsilon_{j_i}$  is negative since  $j_{i+1}$  is to the right of  $j_i$ , so  $\lambda_{j_{i+1}} < \lambda_{j_i}$ .) We call  $j_1, \dots, j_N$  the *first string* through the array. It

is useful to express this by connecting the dots labelled by each pair  $j_i, j_{i+1}$ . In the example the first string is 2, 5, 4, 7, which is depicted by



and

$$f_0 = X_{7,4} + X_{4,5} + X_{5,2}.$$

To complete the description of the generic element  $f$  in  $\mathfrak{n}^- \cap \mathfrak{p}$  we introduce a subgroup  $G_1$  of  $G$ . Let

$$\begin{aligned} V_0 &= \text{span}_{\mathbf{C}}\{e_{j_i} : i = 1, \dots, N\} \\ W_0 &= \text{span}_{\mathbf{C}}\{e_j : j \notin \{j_1, \dots, j_N\}\}. \end{aligned}$$

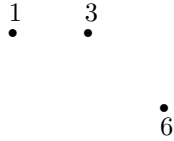
Define

$$G_1 = \{g \in G : g(W_0) \subset W_0 \text{ and } g(e_{j_i}) = e_{j_i}, i = 1, \dots, N\}.$$

Here are a few properties of  $G_1$  that allow the inductive procedure to work.

- (i)  $G_1$  is isomorphic to  $GL(n - N)$ .
- (ii)  $G_1$  is  $\Theta$ -stable and, setting  $K_1 = (G_1)^\Theta = K \cap G_1$ , the pair  $(G_1, K_1)$  is of the same type as  $(G, K)$ .
- (iii)  $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$  is a Cartan subalgebra of  $\mathfrak{g}_1$  (and of  $\mathfrak{k}_1$ ) and  $\Delta_{1,c}^+ := \{\alpha|_{\mathfrak{h}_1} : \alpha \in \Delta_c^+, \mathfrak{g}^{(\alpha)} \subset \mathfrak{g}_1, \alpha|_{\mathfrak{h}_1} \neq 0\}$  is a positive system of roots in  $\mathfrak{k}_1$ .
- (iv)  $\mathfrak{b}_1 := \mathfrak{b} \cap \mathfrak{g}_1$  is the Borel subalgebra of  $\mathfrak{g}_1$  defined by  $\lambda_1 := \lambda|_{\mathfrak{h}_1}$ , which is  $\Delta(\mathfrak{h}_1, \mathfrak{g}_1)$ -regular and  $\Delta_{1,c}^+$ -dominant.  $\mathcal{Q}_1 = K_1 \cdot \mathfrak{b}_1$  is a closed orbit in the flag variety for  $G_1$ .
- (v)  $G_1$  centralizes  $f_0$ .

Working in  $G_1$  we choose  $f_1 \in \mathfrak{n}_1 \cap \mathfrak{p}_1 (= \mathfrak{n} \cap \mathfrak{g}_1 \cap \mathfrak{p})$  in the same way that  $f_0$  was chosen in  $\mathfrak{g}$ . This amounts to omitting the dots of the first string through the array to obtain a smaller array and forming a second string. This second string consists of the labels of the dots farthest to the right in the blocks of the smaller array. In the example the smaller array is



and  $f_1 = X_{6,3}$ . A crucial observation is that in passing to the smaller array it is possible (and likely) that several blocks ‘collapse’ to one block. (In the example, 1 and 3 are in different blocks of the array for  $\mathfrak{g}$ , but are in the same block of the array for  $\mathfrak{g}_1$ .)

Continue by defining a subgroup  $G_2$  of  $G_1$  just as  $G_1$  was chosen in  $G$ . In this way there is a sequence of subgroups  $G = G_0 \supset G_1 \supset \cdots \supset G_m$  and a sequence  $f_i \in \mathfrak{n}^- \cap \mathfrak{p}$ ,  $i = 1, \dots, m-1$ . The procedure ends when  $f_m = 0$  (i.e., when there is at most one block in the array for  $\mathfrak{g}_m$ ). It follows from [3, Thm. 3.2] that

$$f = f_0 + \cdots + f_{m-1} \quad (1.1)$$

is generic in  $\mathfrak{n}^- \cap \mathfrak{p}$ .

*Remark 1.2.* The construction given here differs slightly from that of [3] in that the strings in [3] pass through the dots farthest to the *left* in in each block. One easily checks that there is a  $k \in K$  so that our  $f$  and  $G_i$  are conjugate by  $k$  to the  $f$  and  $G_i$  of [3]. In fact, the element  $k$  may be chosen to represent an element of  $W(K)$ .

To describe  $\gamma_Q^{-1}(f)$  we need to define several subgroups of  $K$  and of  $K_i = K \cap G_i$ ,  $i = 1, \dots, m$ . Consider  $K$  first. Let  $\Pi$  be the set of simple roots in  $\Delta^+$  and set  $S = \Pi \cap \Delta_c^+$ . Define  $\langle S \rangle = \text{span}_{\mathbb{C}}\{S\} \cap \Delta_c$  and

$$\mathfrak{q}_K = \mathfrak{h} + \sum_{\alpha \in \langle S \rangle \cup \Delta_c^+} \mathfrak{g}^{(-\alpha)}.$$

Then  $\mathfrak{q}_K$  is a parabolic subalgebra of  $\mathfrak{k}$ . Let  $Q_K$  denote the corresponding parabolic subgroup of  $K$ . Note that  $\langle S \rangle$  consists of all roots  $\epsilon_j - \epsilon_k$  for which  $j, k$  are labels of dots in the same block. Now define parabolic subgroups  $Q_{i,K}$ ,  $i = 1, \dots, m$  of  $K_i$  in the same manner. It follows from Theorem 4.1 and Equation (4.3) of [3] that

$$\gamma_Q^{-1}(f) = Q_{m,K} \cdots Q_{1,K} Q_K \cdot \mathfrak{b} \subset \mathfrak{B}. \quad (1.3)$$

**1.2. The action of the maximal torus  $H$ .** It is not the case that a maximal torus of  $K$  acts on each irreducible component of a Springer fiber. However, we will see that the maximal torus  $H$  does in fact act on each component associated to a closed  $K$ -orbit in  $\mathfrak{B}$ . In this subsection we establish this fact and give a variant of (1.3) for which the action of  $H$  is more apparent.

**Lemma 1.4.** *Let  $H$  be the diagonal Cartan subgroup. Then*

$$Q_{m,K} \cdots Q_{1,K} Q_K = (H Q_{m,K}) \cdots (H Q_{1,K}) (H Q_K).$$

*Proof.* We show, by induction, that  $Q_{i,K} \cdots Q_{1,K} Q_K = (H Q_{i,K}) \cdots (H Q_{1,K}) (H Q_K)$  for each  $i = 0, 1, \dots, m$ . The  $i = 0$  case is  $Q_K = H Q_K$ , which clearly holds since  $Q_K$  is a parabolic subgroup of  $K$ . Assume that the  $i - 1$  case holds. We may decompose  $H$  into the product of subgroups  $H'$  and  $H''$  as follows. The subgroup  $H'$  consists of

all  $h'$  so that

$$h'(e_j) = \begin{cases} e_j, & \text{if } j \text{ is one of the first } i \text{ strings} \\ z_j e_j, & \text{otherwise} \end{cases}$$

and  $H''$  consists of all  $h''$  so that

$$h''(e_j) = \begin{cases} z_j e_j, & \text{if } j \text{ is one of the first } i \text{ strings} \\ e_j, & \text{otherwise.} \end{cases}$$

(Here  $z_j \in \mathbf{C}$ .) Note that  $H' \subset Q_{i,K}$  and  $H''$  commutes with  $K_i$  (and so commutes with  $Q_{i,K}$ ). Therefore

$$\begin{aligned} (HQ_{i,K}) \cdots (HQ_{1,K})(HQ_K) &= (H'Q_{i,K})(H''HQ_{i-1,K}) \cdots (HQ_{1,K})(HQ_K) \\ &= Q_{i,K}(HQ_{i-1,K}) \cdots (HQ_{1,K})(HQ_K) \\ &= Q_{i,K} \cdots Q_{1,K}Q_K, \text{ by the inductive hypothesis.} \end{aligned}$$

□

**Corollary 1.5.** *The maximal torus  $H$  acts on  $\gamma_Q^{-1}(f)$ .*

**Definition 1.6.** Let  $\mathfrak{q}_i = \mathfrak{h} + \mathfrak{q}_{i,K}$  for  $i = 1, \dots, m$  and  $\mathfrak{q}_0 = \mathfrak{q}_K$ .

It is easy to see that  $\mathfrak{q}_i$  is a parabolic subalgebra of the reductive Lie algebra  $\mathfrak{h} + \mathfrak{k}_i$ . Write  $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i^-$  and write the corresponding parabolic subgroup  $Q_i$  as  $L_i U_i^-$ .

*Remark 1.7.* Note that  $L_i$  is slightly different than the group  $L_i$  in [3] in that we are including the full torus  $H$  in  $L_i$ . Also, the group  $Q_i$  appearing in [3] is some parabolic subgroup of  $G_i$ . We have no need to consider such a group and use  $Q_i$  to denote the group  $HQ_{i,K}$  of [3].

The next proposition follows from Lemma 1.4, equation (1.3) and [3, Theorem 4.8].

**Proposition 1.8.**  $\gamma_Q^{-1}(f) = Q_m \cdots Q_1 Q_0 \cdot \mathfrak{b} = L_m \cdots L_1 L_0 \cdot \mathfrak{b}$ .

**1.3. The parabolics  $Q_i$ .** There are two useful descriptions of the parabolic subgroups  $Q_i$ , one in terms of roots and the other in terms of flags.

Suppose that there are  $N_i$  blocks in the array for  $\mathfrak{g}_i$ . List them (from left to right) as  $B_1^i, \dots, B_{N_i}^i$  (where the  $i = 0$  case is the case of the original array in  $\mathfrak{g}$ ).

Let  $S_i$  be the set of simple roots in  $\Delta(\mathfrak{h}, \mathfrak{g}_i) \cap \Delta^+$  that are compact. Then  $\langle S_i \rangle := \text{span}_{\mathbf{C}}\{S_i\} \cap \Delta_{i,c}^+$  is the set of roots  $\epsilon_j - \epsilon_k$  so that  $j, k$  are both in a block  $B_l^i$ , for some  $l$ . Then

$$\begin{aligned} \mathfrak{q}_i &= \mathfrak{l}_i + \mathfrak{u}_i^- \text{ with} \\ \mathfrak{l}_i &= \mathfrak{h} + \sum_{\alpha \in \langle S_i \rangle} \mathfrak{g}^{(\alpha)} \\ \mathfrak{u}_i^- &= \sum_{\alpha \in \Delta_{i,c}^+ \setminus \langle S_i \rangle} \mathfrak{g}^{(-\alpha)}. \end{aligned}$$

Alternatively, we may describe  $Q_i$  as the stabilizer of a flag. Let  $(F^i)$  be the flag  $\{0\} \subset F_{N_i}^i \subset F_{N_i-1}^i \subset \cdots \subset F_2^i \subset F_1^i$ , with

$$F_k^i = \text{span}_{\mathbb{C}}\{e_j : j \in B_k^i \cup \cdots B_{N_i}^i\}, k = 1, 2, \dots, N_i.$$

Then  $Q_i$  is the subgroup of  $HK_i$  stabilizing the flag  $(F^i)$ .

Here are several immediate properties of  $Q_i = L_i U_i^-$ ,  $i = 0, 1, \dots, m$ .

- (1)  $L_i$  is isomorphic to  $(Torus) \times \Pi_l GL(n_l^i)$  where  $n_l^i$  is the cardinality of  $B_l^i$ .
- (2)  $Q_i$  stabilizes  $f_k$ , for  $k < i$ .
- (3)  $\mathfrak{u}_i^- \subset \mathfrak{u}_{i-1}^-$ ,  $i = 1, 2, \dots, m$ .

The next lemma will be used in a crucial way in Section 2.

**Lemma 1.9.** *For  $i = 1, \dots, m$  the following facts hold.*

- (i)  $L_i \cap Q_{i-1}$  is a parabolic subgroup of  $L_i$ , and  $Q_i \cap Q_{i-1}$  is a parabolic subgroup of  $Q_i$ .
- (ii) The nilradical of  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$  is  $\mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-$ , and a Levi factor of  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$  is  $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$ .
- (iii) The nilradical of  $\mathfrak{q}_i \cap \mathfrak{q}_{i-1}$  is  $\mathfrak{u}_i^- + (\mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-)$ , and a Levi factor is  $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$ .
- (iv) Letting  $\mathfrak{v}_i = \mathfrak{u}_{i-1}^- \cap \mathfrak{l}_i$ ,  $\mathfrak{v}_j \cap \mathfrak{v}_k = \{0\}$ , for  $j \neq k$ .

*Proof.* Since  $L_i \cap Q_{i-1} \supset L_i \cap (B \cap K_{i-1}) = L_i \cap B$ , a Borel subgroup of  $L_i$ , we see that  $L_i \cap Q_{i-1}$  is a parabolic subgroup of  $L_i$ . As the nilradical  $U_i^-$  of  $Q_i$  is contained in  $U_{i-1}^-$  by property (3) above, we see that  $Q_i \cap Q_{i-1}$  contains a Borel subgroup of  $Q_i$ , so  $Q_i \cap Q_{i-1}$  is a parabolic subgroup of  $Q_i$ .

The nilradical of  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$  is spanned by  $\mathfrak{g}^{(-\alpha)}$  for  $\alpha = \epsilon_a - \epsilon_b$ ,  $a < b$ , with  $a$  and  $b$  in the same block for  $\mathfrak{g}_i$ , but in different blocks for  $\mathfrak{g}_{i-1}$ . These are precisely the roots in  $\mathfrak{u}_{i-1}^- \cap \mathfrak{l}_i$ . A Levi factor of  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$  is spanned by  $\mathfrak{h}$  along with the  $\mathfrak{g}^{(\pm\alpha)}$  for  $\alpha = \epsilon_a - \epsilon_b$ ,  $a < b$ , with  $a$  and  $b$  in the same block for  $\mathfrak{g}_i$  and for  $\mathfrak{g}_{i-1}$ . This is  $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$ . This proves (ii); (iii) follows since  $\mathfrak{q}_i \cap \mathfrak{q}_{i-1} = (\mathfrak{l}_i \cap \mathfrak{q}_{i-1}) \oplus \mathfrak{u}_i^-$  and  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1} = \mathfrak{l}_i \cap \mathfrak{l}_{i-1} + \mathfrak{l}_i \cap \mathfrak{u}_{i-1}^-$ .

To verify (iv) we may assume that  $j < k$ . Suppose that  $\mathfrak{v}_j \cap \mathfrak{v}_k$  is nonzero; since  $\mathfrak{v}_j \cap \mathfrak{v}_k$  is  $\mathfrak{h}$ -stable it must contain some root space, say  $\mathfrak{g}^{(-(\epsilon_a - \epsilon_b))} \subset \mathfrak{v}_j \cap \mathfrak{v}_k$ . Then  $a, b$  are in the same block for  $\mathfrak{g}_k$ , but in different blocks for  $\mathfrak{g}_{k-1}$ . But since  $j \leq k-1$ ,  $a$  and  $b$  must be in different blocks for both  $\mathfrak{g}_{j-1}$  and  $\mathfrak{g}_j$ . But this contradicts  $\mathfrak{g}^{(-(\epsilon_a - \epsilon_b))} \subset \mathfrak{v}_j$ .  $\square$

## 2. THE STRUCTURE OF $\gamma_{\mathcal{Q}}^{-1}(f)$

In this section we show that  $\gamma_{\mathcal{Q}}^{-1}(f)$  is isomorphic, as an algebraic variety, to an iterated bundle. Since the iterated bundle is smooth, it follows that  $\gamma_{\mathcal{Q}}^{-1}(f)$  is smooth. Note that we are giving  $\gamma_{\mathcal{Q}}^{-1}(f)$  the reduced scheme structure induced by the closed embedding of  $\gamma_{\mathcal{Q}}^{-1}(f)$  in  $\mathfrak{B}$ .

**Definition 2.1.** Let  $R_i = Q_i \cap Q_{i-1}$ , for  $i = 1, 2, \dots, m$ , and  $R_0 = Q_0 \cap B (= K \cap B)$ .

For  $k = 0, \dots, m$ , consider  $Q_k \times Q_{k-1} \times \dots \times Q_0$  with the mixing action of  $R_k \times R_{k-1} \times \dots \times R_0$  given by

$$(q_k, q_{k-1}, \dots, q_1, q_0) \cdot (r_k, r_{k-1}, \dots, r_1, r_0) \\ = (q_k r_k, r_k^{-1} q_{k-1} r_{k-1}, \dots, r_2^{-1} q_1 r_1, r_1^{-1} q_0 r_0).$$

We denote the quotient by

$$X_k = Q_k \times_{R_k} Q_{k-1} \times_{R_{k-1}} \dots \times_{R_2} Q_1 \times_{R_1} Q_0 / R_0. \quad (2.2)$$

The equivalence class of  $(q_k, \dots, q_1, q_0) \in Q_k \times \dots \times Q_1 \times Q_0$  is denoted by  $[q_k, \dots, q_1, q_0]$ . We will write  $X = X_m$ . The map

$$Q_m \times \dots \times Q_1 \times Q_0 \rightarrow \gamma_{\mathcal{Q}}^{-1}(f) \quad (2.3)$$

defined by  $(q_m, \dots, q_1, q_0) \mapsto q_m \dots q_1 q_0 \cdot \mathfrak{b}$  is a surjection from  $Q_m \times \dots \times Q_1 \times Q_0$  onto  $\gamma_{\mathcal{Q}}^{-1}(f)$  (by Prop. 1.8) and clearly descends to a surjection

$$F : X \rightarrow \gamma_{\mathcal{Q}}^{-1}(f). \quad (2.4)$$

We define  $F_k : X_k \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  by the analogous formula (so  $F = F_m$ ).

**Proposition 2.5.** *The space  $X = Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \dots \times_{R_2} Q_1 \times_{R_1} Q_0 / R_0$  is a smooth projective variety and  $F : X \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is a morphism of varieties.*

*Proof.* This type of argument is fairly standard, but we include it for completeness. We show by induction on  $k$  that  $X_k$  is a smooth projective variety with a  $Q_k$ -action, and that  $F_k : X_k \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is a morphism of varieties. If  $k = 0$ , then  $X_0 = Q_0 / R_0$  is a partial flag variety for  $Q_0$ , since  $R_0$  contains a Borel subgroup of  $Q_0$ . In particular,  $X_0$  is a smooth projective  $Q_0$ -variety. Moreover, since the map  $Q_0 \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is constant on  $R_0$ -orbits, the universal mapping property of quotients ([4, II.6.3]) implies that the induced map  $F_0 : X_0 \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is a morphism of varieties.

Assume that our assertions have been proved for  $X_{k-1}$ . Let  $R_m$  act by the mixing action on  $Q_k \times X_{k-1}$ . Now,  $Q_k \rightarrow Q_k / R_k$  is a principal  $R_k$ -bundle, and  $X_{k-1}$  is projective. Moreover, some power of any line bundle on  $X_{k-1}$  is  $Q_{k-1}$ -equivariant, by [24, Cor. 1.6], so  $X_{k-1}$  has a  $Q_{k-1}$ -equivariant ample line bundle. This line bundle is  $R_k$ -equivariant, as  $R_k \subset Q_{k-1}$ . By [24, Prop. 7.1], this implies the existence of a principal bundle

$$Q_k \times X_{k-1} \rightarrow X_k := Q_k \times_{R_k} X_{k-1},$$

where  $X_k$  is quasi-projective. To see that  $X_k$  is projective, we need to show that  $X_k$  is complete. As in the proof of [24, Prop. 7.1], we have a fiber square

$$\begin{array}{ccc} Q_k \times X_{k-1} & \longrightarrow & Q_k \\ \downarrow & & \downarrow \\ X_k & \longrightarrow & Q_k / R_k. \end{array}$$

Since  $X_{k-1}$  is projective, the top map is proper. The vertical maps are flat and surjective (as they are principal bundle maps), hence faithfully flat. Therefore, since the top map is proper, by descent ([17, Section 8.4-5]), so is the bottom map. As  $Q_k/R_k$  is a partial flag variety (since  $R_k$  contains a Borel subgroup of  $Q_k$ ),  $Q_k/R_k$  is complete. Therefore  $X_k$  is complete. Also, since  $X_{k-1}$  is smooth, the top morphism is smooth; so by descent, the bottom morphism is smooth. As  $Q_k/R_k$  is smooth, we see that  $X_k$  is smooth. Since  $Q_k \times R_k$  acts (algebraically) on  $Q_k \times X_{k-1}$ , by [4, II.6.10],  $Q_k$  acts algebraically on  $X_k$ . Thus,  $X_k$  is a smooth projective variety with a  $Q_k$ -action, as desired.  $\square$

*Remark 2.6.* In the preceding proof, we constructed  $X_k$  inductively as the quotient (in the sense of algebraic geometry, as in [4, II.6]) of  $Q_k \times X_{k-1}$  by  $R_k$ . From this, one can show inductively that  $X_k$  is the quotient (in the same sense) of  $Q_k \times Q_{k-1} \times \cdots \times Q_0$  by  $R_k \times R_{k-1} \cdots \times R_0$ . Indeed, we have a surjective map

$$Q_k \times Q_{k-1} \times \cdots \times Q_0 \rightarrow Q_k \times X_{k-1} \rightarrow X_k.$$

Since  $Q_k \times \cdots \times Q_0$  is irreducible, this implies that  $X_k$  is also. The second map is a quotient by  $R_k$ ; by induction, the first map is a quotient by  $R_{k-1} \times \cdots \times R_0$ . Therefore, the composition is a quotient by  $R_k \times \cdots \times R_0$ . (This follows, for example, because each map is open, being a quotient morphism. Therefore the composition is open; by [4, Lemma II.6.2], the composition is a quotient morphism. The composition is also an orbit map (that is, constant on  $R_k \times \cdots \times R_0$ -orbits). So by definition,  $X_k$  is the quotient of  $Q_k \times Q_{k-1} \times \cdots \times Q_0$  by  $R_k \times R_{k-1} \cdots \times R_0$ .)

*Remark 2.7.* Consider the map

$$Q_m \times Q_{m-1} \times \cdots \times Q_0 \rightarrow K/R_m \times K/R_{m-1} \times \cdots \times K/R_0$$

defined by

$$(q_m, q_{m-1}, \dots, q_0) \mapsto (q_m R_m, q_m q_{m-1} R_{m-1}, \dots, q_m q_{m-1} \cdots q_0 R_0).$$

This map is constant on  $R_m \times \cdots \times R_0$ -orbits, so by the universal mapping property, it induces a map  $\phi : X_m \rightarrow K/R_m \times K/R_{m-1} \times \cdots \times K/R_0$ . If  $V$  is a representation of  $R_m \times \cdots \times R_0$ , there is an induced vector bundle on  $\prod_i K_i/R_i$ . Pulling back by  $\phi$  yields a vector bundle on  $X$  whose sheaf of sections we denote by  $\mathcal{O}_X(V)$ . If  $V$  is a 1-dimensional representation corresponding to a character  $\tau$  we will denote this sheaf simply by  $\mathcal{O}_X(\tau)$ . We will mostly be interested in this when  $V$  is simply a representation of  $R_0$  (that is, for  $i > 0$ ,  $R_i$  acts trivially on  $V$ ).

*Remark 2.8.* The analogues of the preceding proposition and remarks hold for other varieties constructed as quotients by mixing actions. We will use this below.

**Proposition 2.9.** *The map  $F$  is a bijection between  $X$  and  $\gamma_Q^{-1}(f)$ .*

*Proof.* We prove a little more than what is stated. Consider  $X_k$  as in (2.2) and  $F_k : X_k \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  as above. We apply induction on  $k$  to prove that each  $F_k$ ,  $k = 0, 1, \dots, m$ , is a bijection onto its image  $Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$ .

The  $k = 0$  case is immediate since  $X_0 = Q_0/R_0$  and  $R_0 = Q_0 \cap B$ . Suppose  $k \geq 1$  and

$$q_k \cdots q_1 q_0 \cdot \mathfrak{b} = q'_k \cdots q'_1 q'_0 \cdot \mathfrak{b}.$$

Then for some  $b \in B \cap K$ ,

$$q_k^{-1} q'_k = q_{k-1} \cdots q_0 b q'^{-1}_0 \cdots q'^{-1}_{k-1}. \quad (2.10)$$

Claim :  $q_k^{-1} q'_k \in R_k$ .

Once the claim is proved, it will follow that  $q_{k-1} \cdots q_1 q_0 \cdot \mathfrak{b} = r_k q'_{k-1} \cdots q'_1 q'_0 \cdot \mathfrak{b}$ , for some  $r_k \in R_k$ . The inductive hypothesis is that  $F_{k-1}$  is a bijection, so

$$[q_{k-1}, \dots, q_0] = [r_k q'_{k-1}, \dots, q'_0] \in X_{k-1}.$$

Therefore,

$$[q_k, q_{k-1}, \dots, q_0] = [q'_k r_k^{-1}, r_k q'_{k-1}, q'_{k-2}, \dots, q'_0] = [q'_k, q'_{k-1}, \dots, q'_0]$$

in  $X_k$ .

To prove the claim it is enough to show that  $q_k^{-1} q'_k \in Q_{k-1}$ . For this we use the following lemma.

**Lemma 2.11.** *There is a sequence of parabolic subalgebras  $\mathfrak{p}^{(k)}$ ,  $k = 0, 1, \dots, m$  of  $\mathfrak{k}$  so that*

- (i)  $\mathfrak{q} = \mathfrak{p}^{(0)} \subset \mathfrak{p}^{(1)} \subset \cdots \subset \mathfrak{p}^{(m)} = \mathfrak{k}$ , and
- (ii)  $\mathfrak{p}^{(k)} \cap (\mathfrak{h} + \mathfrak{k}_k) = \mathfrak{q}_k$ ,  $k = 0, 1, \dots, m$ .

*Proof of lemma.* For each  $k = 0, 1, \dots, m-1$  consider sets  $C_l^{(k)} \subset \{1, 2, \dots, n\}$  with the following properties.

- (1)  $\{1, 2, \dots, n\}$  is the disjoint union of  $C_1^{(k)}, \dots, C_{J_k}^{(k)}$ .
- (2) Each  $C_l^{(k)}$  consists of consecutive integers and lies either in  $\{1, \dots, p\}$  or in  $\{p+1, \dots, n\}$ .
- (3) Every  $\mathfrak{g}_k$ -block has labels that are contained in exactly one  $C_l^{(k)}$ .
- (4) Each  $C_l^{(k)}$  is the union of  $C_j^{(k-1)}$  for several  $j$ .

Set  $C_1^{(m)} = \{1, \dots, p\}$  and  $C_2^{(m)} = \{p+1, \dots, n\}$ .

It follows from (3) that  $J_0 = N$  (the number of blocks in the original array) and  $C_1^{(0)}, \dots, C_N^{(0)}$  are the original blocks.

We need to establish the existence of a family  $C_l^{(k)}$  that satisfies (1)-(5). Intuitively, each  $C_l^{(k)}$  is the union of all  $C_j^{(k-1)}$  meeting a common  $\mathfrak{g}_k$ -block. This is not quite

the case because (2) must be satisfied. The sets  $C_l^{(k)}$  are not uniquely determined by (1)-(5), but we give one choice below.

Let  $N_k$  be the number of  $\mathfrak{g}_k$ -blocks. Let's list these blocks as  $B_1^{(k)}, \dots, B_{N_k}^{(k)}$  by first listing, from left to right, those blocks  $B_1^{(k)}, \dots, B_{p_k}^{(k)}$  occurring in the upper row. Then continue with the blocks  $B_{p_k+1}^{(k)}, \dots, B_{N_k}^{(k)}$  in the lower row. Define an increasing sequence of integers  $a_0, a_1, \dots, a_{N_k+1}$  by

$$a_i = \begin{cases} 1, & i = 0 \\ \text{the index of the leftmost dot in } B_i^{(k)}, & i = 1, \dots, p_k \\ p + 1, & i = p_k + 1 \\ \text{the index of the leftmost dot in } B_{i-1}^{(k)}, & i = p_k + 2, \dots, N_k + 1 \end{cases} \quad (2.12)$$

Using the notation  $[a, b) = \{r \in \mathbf{Z} : a \leq r < b\}$  we define

$$\begin{aligned} C_i^{(k)} &= [a_i, a_{i+1}), \text{ for } i = 0, 1, \dots, N_k \text{ and} \\ C_{N_k+1}^{(k)} &= [a_{N_k+1}, n], \end{aligned} \quad (2.13)$$

for  $k = 0, 1, \dots, m-1$ . Note that these sets are not necessarily nonempty, even for a given value of  $k$ . For example, often the first dot in the first block is labelled by 1; in this case  $a_1 = 1$ , so  $C_0^{(k)} = \emptyset$ .

Since  $\{1, \dots, p\}$  is the disjoint union of  $C_0^{(k)}, \dots, C_{p_k}^{(k)}$  and  $\{p+1, \dots, n\}$  is the disjoint union of  $C_{p_k+1}^{(k)}, \dots, C_{N_k+1}^{(k)}$ , properties (1) and (2) hold. Property (3) holds since each  $a_j$  is leftmost in a  $\mathfrak{g}_k$ -block.

It remains to show that (4) holds. For this it suffices to show that each  $C_i^{(k-1)}$  is contained in some  $C_j^{(k)}$ . Let  $a'_0, a'_1, \dots$  be the sequence of integers defined in (2.12), but for  $\mathfrak{g}_{k-1}$ -blocks. We need to check that for any  $i = 0, \dots, N_{k-1}$ ,

$$[a'_i, a'_{i+1}) \subset [a_j, a_{j+1}), \text{ for some } j.$$

For this it suffices to show that  $\{a_j\} \subset \{a'_i\}$ . Suppose  $a_j$  is (leftmost) in a  $\mathfrak{g}_k$ -block  $B_j^{(k)}$ . Then  $a_j$  lies in some  $\mathfrak{g}_{k-1}$ -block  $B_i^{(k-1)}$ . We claim that  $a_j$  is leftmost in  $B_i^{(k-1)}$ , so is therefore equal to  $a'_i$ . Suppose not, then  $a'_i < a_j$ , since  $a'_i$  is leftmost in  $B_i^{(k-1)}$ . If the dot labelled by  $a'_i$  were in some  $\mathfrak{g}_k$ -block it would be in  $B_j^{(k)}$  (since  $a'_i$  and  $a_j$  are in the same  $\mathfrak{g}_{k-1}$ -block). However, this would contradict  $a_j$  being leftmost in  $B_j^{(k)}$ . If the dot labelled by  $a'_i$  were not in any  $\mathfrak{g}_k$ -block, then the  $k^{\text{th}}$  string passes through this dot. But this cannot happen either, since the string passes through the rightmost dot in each  $\mathfrak{g}_{k-1}$ -block. This proves the claim and completes the verification of (4).

The existence of sets  $C_l^{(k)}$  satisfying the properties (1)-(5) is now established and we are ready to finish the proof of the lemma.

Define  $S^{(k)} = \{\epsilon_{a+1} - \epsilon_a : a, a+1 \in C_l^{(k)} \text{ for some } l\}$ . By Property (2),  $\langle S^{(k)} \rangle = \{\epsilon_a - \epsilon_b : a, b \in C_l^{(k)}, \text{ for some } l\}$ .

**Definition 2.14.**  $\mathfrak{p}^{(k)} = (\mathfrak{h} + \sum_{\alpha \in \langle S^{(k)} \rangle} \mathfrak{g}^{(\alpha)}) + \sum_{\alpha \in \Delta_c^+ \setminus \langle S^{(k)} \rangle} \mathfrak{g}^{(-\alpha)}$ .

The fact that  $\mathfrak{p}^{(k)}$  is a parabolic subalgebra of  $\mathfrak{k}$  follows from Property (2). Part (i) of the lemma follows from Property (4). The third property implies that for each  $l$ ,  $C_l^{(k)} \cap (\{1, 2, \dots, n\} \setminus \{\text{the first } k \text{ strings}\})$  is a  $\mathfrak{g}_k$ -block (or is empty). Part (ii) of the lemma follows from this.  $\square$

We are now in position to prove that  $q_k^{-1}q'_k \in Q_{k-1}$ . Let  $P^{(k)} = N_K(\mathfrak{p}^{(k)})$ , the parabolic subgroup with Lie algebra  $\mathfrak{p}^{(k)}$ . Since  $Q_0, \dots, Q_{k-1} \subset P^{(k-1)}$ , the right hand side of (2.10) is in  $P^{(k-1)}$ . But the left hand side is in  $HK_k \subset HK_{k-1}$ . We conclude that  $q_k^{-1}q'_k \in P^{(k-1)} \cap HK_{k-1} = Q_{k-1}$ .  $\square$

Our main goal for the remainder of this section is to prove that  $F : X \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is an isomorphism of varieties. We shall use the following general result.

**Lemma 2.15.** *Let  $Z$  be a projective variety and  $\varphi : Z \rightarrow W$  a morphism with finite fibers.*

(a) *If  $\varphi_*\mathcal{O}_Z = \mathcal{O}_W$ , then  $\varphi$  is an isomorphism.*

(b) *Suppose that  $\varphi$  is surjective and  $W$  is reduced. Let  $\mathcal{L}$  be an ample invertible sheaf on  $W$ . The natural map  $H^0(W, \mathcal{L}^n) \rightarrow H^0(Z, \varphi^*\mathcal{L}^n)$  is injective for all  $n$ . If for  $n$  sufficiently large, this map is an isomorphism, then  $\varphi$  is an isomorphism.*

*Proof.* (a) Since  $\varphi$  is quasi-finite and  $Z$  is projective,  $\varphi$  is a finite morphism ([19, Ex. III.11.2]). Since a finite morphism is affine ([19, Ex. II.3.4]), for any affine open set  $U = \text{Spec}(B)$  in  $W$ ,  $\varphi^{-1}(U)$  is affine, so is equal to some  $\text{Spec}(A)$ . Since  $\varphi_*\mathcal{O}_Z = \mathcal{O}_W$ , we have

$$A = \mathcal{O}(\varphi^{-1}(U)) = \varphi_*\mathcal{O}_Z(U) = \mathcal{O}_W(U) = B.$$

Thus  $\varphi$  is an isomorphism on any affine open set  $U$ , so it is an isomorphism.

(b) Because  $\varphi$  is surjective and  $W$  is reduced, the map  $\mathcal{O}_W \rightarrow \varphi_*\mathcal{O}_Z$  is injective (cf. [2, Ex. 1.21]). Thus, there is an exact sequence of coherent sheaves on  $W$ :

$$0 \rightarrow \mathcal{O}_W \rightarrow \varphi_*\mathcal{O}_Z \rightarrow \mathcal{F} \rightarrow 0.$$

Tensoring (over  $\mathcal{O}_W$ ) with  $\mathcal{L}^n$  yields an exact sequence

$$0 \rightarrow \mathcal{L}^n \rightarrow \varphi_*\mathcal{O}_Z \otimes \mathcal{L}^n \rightarrow \mathcal{F} \otimes \mathcal{L}^n \rightarrow 0. \quad (2.16)$$

Because  $\varphi$  is finite, the higher direct image functors  $R^i\varphi_*$  are 0 for  $i > 0$ , so

$$H^i(Z, \varphi^*\mathcal{L}^n) \cong H^i(W, \varphi_*\varphi^*\mathcal{L}^n) \cong H^i(W, \varphi_*\mathcal{O}_Z \otimes \mathcal{L}^n).$$

Hence the exact sequence in cohomology associated to (2.16) implies that  $H^0(W, \mathcal{L}^n) \rightarrow H^0(Z, \varphi^* \mathcal{L}^n)$  is injective for all  $n$ . Since  $\mathcal{L}$  is ample, for  $n$  sufficiently large,  $H^1(W, \mathcal{L}^n)$  vanishes and  $\mathcal{F} \otimes \mathcal{L}^n$  is generated by global sections. We obtain an exact sequence

$$0 \rightarrow H^0(W, \mathcal{L}^n) \rightarrow H^0(Z, \varphi^* \mathcal{L}^n) \rightarrow H^0(W, \mathcal{F} \otimes \mathcal{L}^n) \rightarrow 0.$$

Suppose that the first map is an isomorphism. Then  $H^0(W, \mathcal{F} \otimes \mathcal{L}^n) = 0$ . Since  $\mathcal{F} \otimes \mathcal{L}^n$  is generated by global sections, we conclude that  $\mathcal{F} \otimes \mathcal{L}^n = 0$ . Since  $\mathcal{L}^n$  is invertible this implies that  $\mathcal{F} = 0$ , so  $\varphi_* \mathcal{O}_Z = \mathcal{O}_W$ . By (a), this implies that  $\varphi$  is an isomorphism.  $\square$

Let  $\tau \in \mathfrak{h}^*$  correspond to a character  $\chi_\tau$  of  $H$ ; extend  $\chi_\tau$  to a character of  $K \cap B = R_0$ . The closed  $K$ -orbit  $\mathcal{Q}$  may be identified with  $K/(K \cap B)$ , so there is an invertible sheaf  $\mathcal{O}_{\mathcal{Q}}(\tau)$  on  $\mathcal{Q}$  (cf. Remark 2.7). Let  $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$  denote the pullback of this sheaf to the subvariety  $\gamma_{\mathcal{Q}}^{-1}(f)$  of  $\mathcal{Q}$ . Likewise, there is an invertible sheaf  $\mathcal{O}_X(\tau)$ , and  $F^* \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau) = \mathcal{O}_X(\tau)$ .

Let  $\tau$  be  $\Delta_c^+$ -dominant and integral. Let  $W_{-\tau}$  be the irreducible finite-dimensional representation of  $K$  of lowest weight  $-\tau$ , and let  $w_{-\tau}$  denote a lowest weight vector in  $W_{-\tau}$ . The Borel-Weil Theorem states that  $W_{-\tau}^* \cong H^0(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\tau))$ . Define

$$p(\tau) = \dim(\text{span}_{\mathbf{C}}\{L_m \cdots L_2 L_1 L_0 \cdot w_{-\tau}\}). \quad (2.17)$$

We will need the following results from [3, Section 6] about  $p(\tau)$ . If  $\tau$  is sufficiently dominant, then

$$p(\tau) = \dim H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)).$$

Moreover,  $p(\tau)$  can be computed inductively, as follows. Write  $U_{-\tau} = \text{span}_{\mathbf{C}}\{L_0 w_{-\tau}\}$ , the irreducible representation of  $L$  having lowest weight  $-\tau$ . Decompose  $U_{-\tau}$  as a representation of  $L_1 \cap L_0$ . Write this decomposition as

$$U_{-\tau}|_{L_1 \cap L_0} = \sum E_{-\tau_i}, \quad (2.18)$$

and let  $w_{-\tau_i}$  denote a lowest weight vector in  $E_{-\tau_i}$ . Then  $F_i := \text{span}_{\mathbf{C}}\{K_1 w_{-\tau_i}\}$  is the irreducible  $K_1$ -representation of lowest weight  $-\tau_i$ . Set

$$p_1(\tau_i) = \dim(\text{span}_{\mathbf{C}}\{L_m \cdots L_2 L_1 \cdot w_{-\tau_i}\}).$$

Thus,  $p_1(\tau_i)$  is the analogue of  $p(\tau)$  for the pair  $(G_1, K_1)$ , with  $\tau_i$  in place of  $\tau$ . We have

$$p(\tau) = \sum_i p_1(\tau_i). \quad (2.19)$$

For arbitrary  $\tau$  (not necessarily dominant), we define

$$q(\tau) = \chi(X, \mathcal{O}_X(\tau)).$$

Here  $\chi$  denotes the Euler characteristic (alternating sum of dimensions of cohomology groups).

The following theorem is the main result of this paper.

**Theorem 2.20.** *The map  $F : X \rightarrow \gamma_{\mathcal{Q}}^{-1}(f)$  is an isomorphism of varieties.*

*Proof.* If  $\tau \in \mathfrak{h}^*$  is  $\Delta_c^+$ -dominant, then  $\mathcal{O}_{\mathcal{Q}}(\tau)$  is ample, so its restriction  $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$  to  $\gamma_{\mathcal{Q}}^{-1}(f)$  is ample. As the pullback of an ample invertible sheaf under a finite morphism is ample, and  $F$  is a finite morphism (cf. Lemma 2.15), the sheaf  $F^*\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau) = \mathcal{O}_X(\tau)$  is ample as well.

By Lemma 2.15, there is an injective map

$$H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)) \rightarrow H^0(X, \mathcal{O}_X(\tau)). \quad (2.21)$$

If  $\tau$  is a sufficiently large multiple of a  $\Delta_c^+$ -dominant character, then since  $\mathcal{O}_X(\tau)$  is ample, the cohomology groups  $H^i(X, \mathcal{O}_X(\tau))$  vanish for  $i > 0$ , and so  $q(\tau) = \dim H^0(X, \mathcal{O}_X(\tau))$ . For such  $\tau$  we also have  $p(\tau) = \dim H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$ . By Lemma 2.15, to prove the theorem it suffices to prove that for any  $\tau$  that is a sufficiently large multiple of a  $\Delta_c^+$ -dominant character,

$$p(\tau) = q(\tau). \quad (2.22)$$

In fact, we will prove that this equation holds for any  $\Delta_c^+$ -dominant  $\tau$ .

Define

$$X(1) = Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \cdots \times Q_1/R'_1,$$

where  $R'_1 = Q_1 \cap B \subset R_1 = Q_1 \cap Q_0$ . Note that  $R'_1 = R_1 \cap B$  is a Borel subgroup of  $R_1$ . Observe that  $X(1)$  is a variety of the same type as  $X$ , but constructed using the pair  $(G_1, K_1)$ . Let

$$q_1(\tau_i) = \chi(X(1), \mathcal{O}_{X(1)}(\tau_i)).$$

To prove (2.22), we proceed by induction on the number of strings  $m$ . If  $m = 0$  then  $\gamma_{\mathcal{Q}}^{-1}(f) = \mathcal{Q}$  and the result holds. Suppose the result holds in case there are  $m - 1$  strings. The inductive hypothesis implies that  $p_1(\tau_i) = q_1(\tau_i)$ . As  $p(\tau) = \sum_i p_1(\tau_i)$  by (2.19), to prove that  $p(\tau) = q(\tau)$ , it suffices to prove that

$$q(\tau) = \sum q_1(\tau_i). \quad (2.23)$$

Consider the diagram

$$\begin{array}{ccc} X(1) = Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \cdots \times Q_1/R'_1 & & \\ \downarrow h & & \\ X \xrightarrow{g} M = Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \cdots \times Q_1/R_1 & & \end{array}$$

Here  $g$  is the map that forgets the last factor, and  $h$  is the projection induced by the inclusion  $R'_1 \subset R_1$ . Both  $g$  and  $h$  are fiber bundles associated to the  $R_1$ -principal bundle

$$\widetilde{M} \rightarrow Q_m \times_{R_m} Q_{m-1} \times_{R_{m-1}} \cdots \times Q_1 \rightarrow M. \quad (2.24)$$

The fibers are isomorphic to the  $R_1$ -varieties  $Q_0/R_0$  and  $R_1/R'_1$ , respectively.

The push-forward sheaf  $R^i g_* \mathcal{O}_X(\tau)$  is the sheaf of sections on  $M$  associated to the  $R_1$ -module  $H^i(Q_0/R_0, \mathcal{O}_{Q_0/R_0}(\tau))$ . Since  $L_0$  is a Levi factor of  $Q_0$ , and  $R_0 = B \cap K$  is a Borel subgroup of  $Q_0$ , there is an isomorphism  $L_0/(L_0 \cap B) \cong Q_0/R_0$ . Therefore,  $H^i(Q_0/R_0, \mathcal{O}_{Q_0/R_0}(\tau)) \cong H^i(L_0/(L_0 \cap B), \mathcal{O}_{L_0/(L_0 \cap B)}(\tau))$ . By the Borel-Weil-Bott Theorem, this group is 0 for  $i > 0$ . For  $i = 0$ , the group is a  $Q_0$ -module whose restriction to  $L_0$  is isomorphic to  $U_{-\tau}^*$ . We use the same notation  $U_{-\tau}^*$  for the restriction of this module to  $R_1 \subset Q_0$ . Thus, we have shown that

$$g_* \mathcal{O}_X(\tau) = \mathcal{O}_M(U_{-\tau}^*), \quad (2.25)$$

and that for  $i > 0$ ,

$$R^i g_* \mathcal{O}_X(\tau) = 0. \quad (2.26)$$

The preceding discussion and the Leray spectral sequence imply that for all  $i$ ,

$$H^i(X, \mathcal{O}_X(\tau)) \cong H^i(M, \mathcal{O}_M(U_{-\tau}^*)). \quad (2.27)$$

Hence

$$q(\tau) = \chi(X, \mathcal{O}_X(\tau)) = \chi(M, \mathcal{O}_M(U_{-\tau}^*)). \quad (2.28)$$

By Lemma 1.9,  $L_1 \cap L_0$  is a Levi factor of  $R_1$ . Thus,  $U_{-\tau}^*$  has a filtration whose associated graded module is a representation on which the unipotent radical of  $R_1$  acts trivially and on which  $L_1 \cap L_0$  acts as  $\oplus E_{-\tau_i}^*$  (cf. (2.18)). This induces a corresponding filtration on the sheaf  $\mathcal{O}_M(U_{-\tau}^*)$ , and since the Euler characteristic is additive with respect to exact sequences, we see that

$$\chi(M, \mathcal{O}_M(U_{-\tau}^*)) = \sum_i \chi(M, \mathcal{O}_M(E_{-\tau_i}^*)). \quad (2.29)$$

Each  $\tau_i$  defines a character of  $R'_1$ , and the push-forward sheaf  $R^i h_* \mathcal{O}_{X(1)}(\tau_i)$  is the sheaf of sections on  $M$  associated to the  $R_1$ -module  $H^i(R_1/R'_1, \mathcal{O}_{R_1/R'_1}(\tau_i))$ . Since  $L_1 \cap L_0$  is a Levi factor of  $R_1$ , and  $R'_1 = R_1 \cap B$  is a Borel subgroup of  $R_1$ , there is an isomorphism  $(L_1 \cap L_0)/(L_1 \cap L_0 \cap B) \cong R_1/R'_1$ . Therefore,

$$H^i(R_1/R'_1, \mathcal{O}_{R_1/R'_1}(\tau_i)) \cong H^i((L_1 \cap L_0)/(L_1 \cap L_0 \cap B), \mathcal{O}_{(L_1 \cap L_0)/((L_1 \cap L_0 \cap B))}(\tau_i)).$$

By the Borel-Weil-Bott Theorem, this group is 0 for  $i > 0$ . For  $i = 0$ , the group is a  $R_1$ -module whose restriction to  $L_1 \cap L_0$  is isomorphic to  $E_{-\tau_i}^*$ . Moreover, we claim that the unipotent radical of  $R_1$  acts trivially on the module  $H^0(R_1/R'_1, \mathcal{O}_{R_1/R'_1}(\tau_i))$ . Indeed, if we denote this unipotent radical by  $N$ , then since  $N$  is unipotent and normal in  $R_1$ , the space of  $N$ -fixed vectors in this module is nonzero and  $R_1$ -stable. Since the  $R_1$ -module  $H^0(R_1/R'_1, \mathcal{O}_{R_1/R'_1}(\tau_i))$  is irreducible (as its restriction to  $L_1 \cap L_0$  is), the space of  $N$ -fixed vectors must be the entire module, proving the claim. Thus, we have shown that

$$h_* \mathcal{O}_{X(1)}(\tau_i) = \mathcal{O}_M(E_{-\tau_i}^*), \quad (2.30)$$

and that for  $i > 0$ ,

$$R^i h_* \mathcal{O}_{X(1)}(\tau_i) = 0. \quad (2.31)$$

Again using the Leray spectral sequence we see that for all  $i$  and all  $\tau_j$ ,

$$H^i(X(1), \mathcal{O}_{X(1)}(\tau_j)) \cong H^i(M, \mathcal{O}_M(U_{-\tau_j}^*)). \quad (2.32)$$

Hence

$$q_1(\tau_i) = \chi(X(1), \mathcal{O}_{X(1)}(\tau_i)) = \chi(M, \mathcal{O}_M(U_{-\tau_i}^*)). \quad (2.33)$$

Equations (2.28), (2.29) and (2.33) imply that  $q(\tau) = \sum q_1(\tau_i)$ , proving the theorem.  $\square$

The next result is a vanishing theorem for the higher cohomology groups of invertible sheaves on  $\gamma_{\mathcal{Q}}^{-1}(f)$  associated to dominant weights.

**Theorem 2.34.** *If  $\tau \in \mathfrak{h}^*$  is  $\Delta_c^+$ -dominant and integral, then for all  $i > 0$ , we have  $H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)) = 0$ .*

*Proof.* We retain the notation of the preceding proof. The proof is by induction on the number of strings  $m$ . If  $m = 0$  then  $\gamma_{\mathcal{Q}}^{-1}(f) = \mathcal{Q}$  and the result holds by the Borel-Weil-Bott theorem. Suppose the result holds in case there are  $m - 1$  strings. We wish to show that  $H^i(X, \mathcal{O}_X(\tau)) = 0$  for  $i > 0$ . By (2.27), this is equivalent to showing that  $H^i(M, \mathcal{O}_M(U_{-\tau}^*)) = 0$  for  $i > 0$ . By the proof of Theorem 2.20, the sheaf  $\mathcal{O}_M(U_{-\tau}^*)$  has a filtration with subquotients isomorphic to  $\mathcal{O}_M(E_{-\tau_j}^*)$ . Induction and the long exact sequence in cohomology imply that our desired vanishing will follow if we can show that  $H^i(M, \mathcal{O}_M(E_{-\tau_j}^*)) = 0$  for  $i > 0$ . By (2.32), this is equivalent to showing that  $H^i(X(1), \mathcal{O}_{X(1)}(\tau_j)) = 0$  for  $i > 0$ . But  $X(1)$  is a component of a Springer fiber associated to a closed  $K_1$ -orbit for the pair  $(G_1, K_1)$ , and with  $m - 1$  strings. Moreover, in the course of the proof of [3, Theorem 6.6], it is proved that each  $\tau_j$  is dominant with respect to the positive system  $\Delta_{1,c}^+$ . The inductive hypothesis implies that  $H^i(X(1), \mathcal{O}_{X(1)}(\tau_j)) = 0$  for  $i > 0$ , completing the proof.  $\square$

**2.1. Some topological consequences of Theorem 2.20.** A consequence of the theorem is that any component of a Springer fiber associated to a closed  $K$ -orbit is a fiber bundle over a generalized flag variety for a (smaller) general linear group having fiber that is a component of a Springer fiber associated to a closed orbit for a smaller pair  $(G', K')$ . To make this precise, let  $S'$  be the set of all labels of dots in the array that are contained in one of the  $m$  strings. Set  $U = \text{span}_{\mathbb{C}}\{e_i : i \in S'\}$  and

$$G' = \{g \in G : g(U) \subset U \text{ and } g(e_j) = e_j, \text{ when } j \notin S'\}$$

Let  $K' = K \cap G'$ ,  $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$  and  $\lambda' = \lambda|_{\mathfrak{h}'}$ . If  $\mathcal{Q}'$  is the corresponding closed  $K'$ -orbit in the flag variety for  $G'$ , then Theorem 2.20 tells us that  $X_{m-1} \simeq \gamma_{\mathcal{Q}'}^{-1}(f)$ . (Note that the algorithm gives the same generic element  $f$ .) This proves the following corollary.

**Corollary 2.35.** *In the setting of the Theorem, there is a fibration*

$$X \rightarrow Q_m/R_m \tag{2.36}$$

having fiber  $X_{m-1}$ . Here  $X_{m-1}$  is a component of a Springer fiber for the smaller pair  $(G', K')$  associated to a closed  $K'$ -orbit in  $\mathfrak{B}'$ , and  $Q_m/R_m$  is a generalized flag variety for  $L_m$ .

We remark that often  $Q_m/R_m$  is just a point (so  $X = X_{m-1}$ ). In this case one may take  $S'$  to be the labels of the dots in the first  $m-1$  strings. Then there is a fibration

$$X \rightarrow Q_{m-1}/R_{m-1}$$

and the fiber is  $X_{m-2}$ , which is again a component of a Springer fiber for the smaller pair  $(G', K')$  associated to a closed  $K'$ -orbit in  $\mathfrak{B}'$ .

If  $Z$  is a topological space with finite-dimensional rational cohomology, we define the Poincaré polynomial of  $Z$  to be

$$P_t(Z) = \sum_i \dim H^i(Z, \mathbf{Q}) t^i.$$

As an application of Theorem 2.20, we obtain the Poincaré polynomial of  $\gamma_{\mathcal{Q}}^{-1}(f)$ .

**Corollary 2.37.** *The variety  $\gamma_{\mathcal{Q}}^{-1}(f)$  is simply connected, and the cohomology ring  $H^*(\gamma_{\mathcal{Q}}^{-1}(f); \mathbf{Z})$  is torsion-free and vanishes in odd dimensions. The Poincaré polynomial of  $\gamma_{\mathcal{Q}}^{-1}(f)$  is*

$$P_t(\gamma_{\mathcal{Q}}^{-1}(f)) = \prod_{i=0}^m P_t(Q_i/R_i) = \prod_{i=0}^m P_t(L_i/(L_i \cap R_i)).$$

*Proof.* We use the notation of (2.2), so  $X = X_m$ . We prove by induction on  $k$  that the cohomology  $H^*(X_k; \mathbf{Z})$  is torsion-free and vanishes in odd dimensions, and

$$P_t(X_k) = \prod_{i=0}^k P_t(Q_i/R_i).$$

For  $k=0$  the result holds since  $X_0 = Q_0/R_0 = L_0/(L_0 \cap B)$  is a flag variety for the reductive group  $L_0$ . Suppose the proposition holds for  $X_{k-1}$ . As in Corollary 2.35, there is a fiber bundle  $X_k \rightarrow Q_k/R_k$  with fiber  $X_{k-1}$ . Since  $Q_k/R_k$  and  $X_{k-1}$  are simply connected, the long exact sequence for homotopy implies that  $X_k$  is simply connected. Because  $H^*(X_{k-1}; \mathbf{Z})$  is free and  $Q_k/R_k$  is simply connected, the Leray spectral sequence for the cohomology of this fiber bundle has  $E_2$  term

$$E_2 = H^*(Q_k/R_k; \mathbf{Z}) \otimes H^*(X_{k-1}; \mathbf{Z}).$$

Since base and fiber have no odd-dimensional cohomology, the spectral sequence degenerates at  $E_2$ . This implies that as  $\mathbf{Z}$ -modules,  $H^*(X_k; \mathbf{Z})$  and  $H^*(Q_k/R_k; \mathbf{Z}) \otimes$

$H^*(X_{k-1}; \mathbf{Z})$  are isomorphic, so

$$P_t(X_k) = P_t(X_{k-1})P_t(Q_k/R_k) = \prod_{i=0}^k P_t(Q_i/R_i),$$

proving the first equality of the proposition. The second equality holds because  $L_i$  is a Levi factor of  $Q_i$ .  $\square$

*Remark 2.38.* The polynomials  $P_t(L_i/(L_i \cap R_i))$  can be easily computed from the array by the following procedure. If  $M$  is a reductive group, write  $p_M$  for the Poincaré polynomial of the flag variety for  $M$ . Since all odd-dimensional cohomology vanishes, it is convenient to write  $u = t^2$ . If  $n \geq 2$ , then

$$p_{GL(n)} = \frac{(1-u^2)(1-u^3) \cdots (1-u^n)}{(1-u)^{n-1}};$$

$p_{GL(1)} = 1$ . If  $P$  is a parabolic subgroup of  $M$ , with Levi factor  $M'$ , then  $P_t(M/P) = \frac{p_M}{p_{M'}}$ , as follows by considering the fibration  $M/B_M \rightarrow M/P$  with fibers  $P/B_M$  (here  $B_M \subset P$  is a Borel subgroup of  $M$ ). By Lemma 1.9,  $L_i \cap L_{i-1}$  is a Levi factor of  $L_i \cap R_i$ , so if  $i > 0$ ,

$$P_t(L_i/(L_i \cap R_i)) = \frac{p_{L_i}}{p_{L_i \cap L_{i-1}}},$$

and  $P_t(L_0/(L_0 \cap R_0)) = p_{L_0}$ . Recall that we denoted the blocks in the array for  $\mathfrak{g}_i$  as  $B_1^i, \dots, B_{N_i}^i$ , and let  $n_l^i$  equal the cardinality of  $B_l^i$ . By the discussion before Lemma 1.9,

$$p_{L_i} = \prod_l p_{GL(n_l^i)}.$$

We define subblocks of the blocks for  $\mathfrak{g}_i$  (in case  $i > 0$ ) as follows. Let  $a$  and  $b$  be in the same block for  $\mathfrak{g}_i$ . If they are also in the same block for  $\mathfrak{g}_{i-1}$ , we say they are in the same subblock. List the lengths of the subblocks as  $m_1^i, m_2^i, \dots$ . By the proof of Lemma 1.9, the roots of  $\mathfrak{l}_i \cap \mathfrak{l}_{i-1}$  are  $\epsilon_a - \epsilon_b$ ,  $a < b$ , with  $a$  and  $b$  in the same subblock. Hence

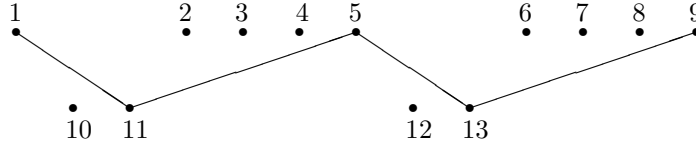
$$p_{L_i \cap L_{i-1}} = \prod_l p_{GL(m_l^i)}.$$

From these facts one can readily calculate  $P_t(\gamma_{\mathcal{Q}}^{-1}(f))$ . For example, in the case of the array considered in Section 1.1, we find that  $P_t(\gamma_{\mathcal{Q}}^{-1}(f)) = (1+u)^4 = (1+t^2)^4$ .

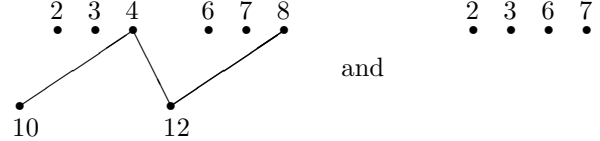
*Example 2.39.* We illustrate these ideas with the following example. Consider the pair  $(GL(14), GL(9) \times GL(4))$  and

$$\lambda = (13, 10, 9, 8, 7, 4, 3, 2, 1 \mid 12, 11, 6, 5).$$

The array with first string is



The arrays for  $(G_1, K_1)$  and  $(G_2, K_2)$  are



Then (except for a few factors of the torus, which play no role)

$$L = GL(1) \times GL(4) \times GL(4) \times GL(2) \times GL(2)$$

$$L_1 = GL(3) \times GL(3) \times GL(1), \quad L_1 \cap L = L_1$$

$$L_2 = GL(4), \quad L_2 \cap L_1 = GL(2) \times GL(2).$$

From this it is easy to see that the Poincaré polynomial is

$$(1 + u)^3(1 + u + u^2)^3(1 + u + u^2 + u^3)^3, \quad u = t^2.$$

The generalized flag variety  $Q_m/R_m$  ( $m = 2$ ) is the Grassmannian  $G_2(\mathbf{C}^4)$ . Therefore, the fibration (2.36) is

$$X_1 \rightarrow G_2(\mathbf{C}^4),$$

and  $X_1$  is the component of a Springer fiber for  $(GL(9), GL(5) \times GL(4))$  with

$$\lambda' = (13, *, *, 8, 7, *, *, 2, 1 \mid 12, 11, 6, 5).$$

Here the coordinates in the 2, 3, 6 and 7 places are omitted in passing to  $G'$ .

### 3. REPRESENTATIONS ON TANGENT SPACES TO ITERATED BUNDLES AT FIXED POINTS

The maximal torus  $H$  acts on  $\gamma_Q^{-1}(f)$  with finitely many fixed points, which we describe explicitly in the next section. If  $x \in \gamma_Q^{-1}(f)$  is an  $H$ -fixed point, then the tangent space  $T_x \gamma_Q^{-1}(f)$  is a representation of  $H$ . Using the structure of  $\gamma_Q^{-1}(f)$  as an iterated bundle, we can describe the weights of this representation. Then we can apply localization theorems in equivariant cohomology and  $K$ -theory to the study of  $\gamma_Q^{-1}(f)$ .

In this section we will describe the representations on tangent spaces to fixed points in case  $H$  is any algebraic group acting on any variety  $X$  constructed as an iterated bundle. Related calculations have been done, for example, in the case of Bott-Samelson-Demazure varieties (see e.g. [5]). However, for lack of a reference, we have decided to give the result for a general iterated bundle. In the next section, we will apply these results to the case where  $H$  is a maximal torus acting on  $X = \gamma_Q^{-1}(f)$ .

**3.1. Generalities.** We begin with some notation. Given a left action of an algebraic group  $Q$  on a space  $X$  and an element  $q \in Q$ , let  $L(q) : X \rightarrow X$  denote the map induced by the left action of  $q$  and let  $C(q)$  denote the map  $Q \rightarrow Q$  given by conjugation by  $q$ . If  $H$  is a subgroup of  $Q$  and  $x \in X$  is  $H$ -fixed, then the map  $H \rightarrow GL(T_x X)$  given by  $h \mapsto L(h)_* = dL(h)$  defines a representation of  $H$ . Finally, if  $\xi \in \mathfrak{q}$  and  $X$  is smooth, let  $\xi^\#$  denote the induced vector field on  $X$ , whose value  $\xi_x^\#$  at  $x \in X$  is determined by the rule that if  $\phi$  is a function on  $X$ , then

$$\xi_x^\# \phi = \frac{d}{dt}(\phi(\exp(t\xi)x))|_{t=0}.$$

**Lemma 3.1.** *Let  $x \in X$  be fixed by  $q \in Q$  and let  $\xi \in \mathfrak{q}$ . Then*

$$L(q)_*(\xi_m^\#) = (\text{Ad}(q)\xi)_m^\#. \quad (3.2)$$

*Proof.* If we apply the left side of (3.2) to a function  $\phi$  on  $X$ , we obtain

$$\frac{d}{dt}(\phi(q \exp(t\xi)x))|_{t=0}.$$

If we apply the right hand side of (3.2) to  $\phi$ , we obtain

$$\frac{d}{dt}(\phi(\exp(t \text{Ad}(q)\xi)x))|_{t=0} = \frac{d}{dt}(\phi(q \exp(t\xi)q^{-1}x))|_{t=0}.$$

Since  $x$  is fixed by  $q^{-1}$ , the two calculations agree.  $\square$

**3.2. The case of a mixed space.** Throughout this subsection,  $Q$  will denote an algebraic group, and  $H$  and  $R$  will denote subgroups of  $Q$ . Suppose that  $M$  is a smooth algebraic variety with an  $R$ -action. We write  $\tilde{X} = Q \times M$  and we let  $X$  denote the “mixed space”  $X = Q \times_R M$ . Under mild hypotheses (cf. Remark 2.8)  $X$  is a smooth algebraic variety, and we assume this is the case. Let  $\pi : \tilde{X} \rightarrow X$  denote the quotient morphism. We write  $[q, m]$  for  $\pi(q, m)$  (for  $q \in Q$ ,  $m \in M$ ). The group  $Q \times R$  acts on  $\tilde{X}$ , so for  $(q, r) \in Q \times R$  we have maps  $L(q, r) := L(q) \times L(r)$  and  $C(q) \times L(r)$  from  $\tilde{X}$  to  $\tilde{X}$ . Also,  $Q$  acts on  $X$ , and we have  $\pi \circ L(q, 1) = L(q) \circ \pi$ . Note also that if  $\xi \in \mathfrak{r}$  then there is an induced vector field  $\xi^\#$  on  $M$ .

Observe that  $L(q, 1)_*$  maps  $V = \mathfrak{q} \times T_m M$  isomorphically onto  $T_{(q, m)} \tilde{X}$ . We define  $\rho : V \rightarrow T_{[q, m]} X$  as the composition

$$\rho = \pi_* \circ L(q, 1)_*. \quad (3.3)$$

**Lemma 3.4.** *The point  $[q, m] \in X$  is  $H$ -fixed  $\Leftrightarrow q^{-1}Hq \subset R$ , and  $m$  is  $q^{-1}Hq$ -fixed.*

*Proof.* This is a straightforward computation.  $\square$

Suppose now that  $[q, m]$  is  $H$ -fixed. Let  $V = \mathfrak{q} \oplus T_m M$ , and define an  $H$ -module structure on  $V$  by the formula

$$h \cdot (\xi, v) = (\text{Ad}(q^{-1}h)\xi, L(q^{-1}h)_*v),$$

for  $\xi \in \mathfrak{q}$ ,  $v \in T_m M$ . We define an  $H$ -module structure on  $\mathfrak{r}$  by the formula

$$h \cdot \xi = \text{Ad}(q^{-1}hq)\xi.$$

**Lemma 3.5.** *The embedding  $\psi : \mathfrak{r} \rightarrow V$  defined by  $\psi(\xi) = (\xi, -\xi_m^\#)$  is  $H$ -equivariant.*

*Proof.* Write  $s = q^{-1}hq$ . Using the preceding two lemmas, we have

$$\psi(h \cdot \xi) = (\text{Ad}(s)\xi, -(\text{Ad}(s)\xi)_m^\#) = (\text{Ad}(s)\xi, -L(s)_*(\xi_m^\#)) = h \cdot \psi(\xi),$$

as desired.  $\square$

Let  $V_1$  denote the  $H$ -submodule  $\psi(\mathfrak{r})$  of  $V$ .

The main purpose of this subsection is to prove the following proposition.

**Proposition 3.6.** *With notation as above, assume that  $[q, m]$  is an  $H$ -fixed point of  $X$ . The map  $\rho : V \rightarrow T_{[q, m]}X$  is  $H$ -equivariant with kernel  $V_1$ , and hence induces an  $H$ -module isomorphism  $V/V_1 \rightarrow T_{[q, m]}X$ .*

*Proof.* To show that  $\rho$  is  $H$ -equivariant we must show that

$$L(h)_*\rho(\xi, v) = \rho(\text{Ad}(s)\xi, L(s)_*v), \quad (3.7)$$

where  $s = q^{-1}hq$ . Now,

$$L(h)_* \circ \rho = L(h)_* \circ \pi_* \circ L(q, 1)_* = (\pi \circ L(hq, 1))_*.$$

Direct computation shows that

$$\pi \circ L(hq, 1) = \pi \circ L(q, 1) \circ (C(s) \times L(s)).$$

Therefore,

$$L(h)_* \circ \rho = \pi_* \circ L(q, 1)_* \circ (C(s)_* \times L(s)_*) = \rho \circ (\text{Ad}(s) \times L(s)_*),$$

which implies that  $\rho$  is  $H$ -equivariant.

Since  $V_1$  and  $\ker \rho$  have the same dimension, to show that they are equal it suffices to show that  $V_1 \subset \ker \rho$ . Observe that for all  $r \in R$ , the point  $(qr, r^{-1}m)$  lies in the fiber  $\pi^{-1}([q, m])$ . This implies that the vector  $v \in T_{(q, m)}\tilde{X}$  defined by

$$v \cdot \phi = \frac{d}{dt}(\phi(q \exp(t\xi), \exp(-t\xi)m))|_{t=0}$$

satisfies  $\pi_*(v) = 0$ . But

$$v = L(q, 1)_*(\xi, -\xi_m^\#),$$

so we see that  $\rho((\xi, -\xi_m^\#)) = 0$ . Hence  $V_1 \subset \ker \rho$ , as desired.  $\square$

Let  $H$  act on  $\mathfrak{q}$  by the rule

$$h \cdot \xi = \text{Ad}(q^{-1}hq)\xi.$$

If  $[q, m]$  is  $H$ -fixed, then  $q^{-1}Hq \subset R$ . Thus,  $\mathfrak{r}$  is an  $H$ -submodule of  $\mathfrak{q}$ , and hence  $\mathfrak{q}/\mathfrak{r}$  is an  $H$ -module. Moreover,  $m$  is  $q^{-1}Hq$ -fixed, so so the map  $H \rightarrow GL(T_m M)$

given by  $h \mapsto L(q^{-1}hq)_*$  defines an  $H$ -module structure on  $T_m M$ . By combining these  $H$ -module structures, we obtain an  $H$ -module structure on  $\mathfrak{q}/\mathfrak{r} \oplus T_m M$ . On the other hand, since  $[q, m]$  is  $H$ -fixed, there is an  $H$ -module structure on  $T_{[q, m]} X$ .

**Corollary 3.8.** *Assume that the hypotheses of the preceding proposition hold, and assume in addition that  $H$  is reductive. Then  $T_{[q, m]} X$  and  $\mathfrak{q}/\mathfrak{r} \oplus T_m M$  are isomorphic  $H$ -modules.*

*Proof.* Let  $W$  be an irreducible  $H$ -module. If  $U$  is any  $H$ -module let  $\text{mult}_W(U)$  denote the multiplicity of  $W$  in  $U$ . By the preceding proposition, we have

$$\text{mult}_W(T_{[q, m]} X) = \text{mult}_W(V) - \text{mult}_W(V_1).$$

Since  $V = \mathfrak{q} \oplus T_m M$ , and since by definition,  $V_1 \cong \mathfrak{r}$  as  $H$ -modules, we have

$$\begin{aligned} \text{mult}_W(T_{[q, m]} X) &= \text{mult}_W(\mathfrak{q}) + \text{mult}_W(T_m M) - \text{mult}_W(\mathfrak{r}) \\ &= \text{mult}_W(\mathfrak{q}/\mathfrak{r}) + \text{mult}_W(T_m M). \end{aligned}$$

Since any irreducible  $H$ -module occurs with the same multiplicity in  $T_{[q, m]} X$  and in  $\mathfrak{q}/\mathfrak{r} \oplus T_m M$ , we conclude that these  $H$ -modules are isomorphic.  $\square$

**3.3. Iterated bundles.** We now apply the results of the preceding subsection to iterated bundles. Let  $Q_0, \dots, Q_n$  be subgroups of an algebraic group  $G$  and suppose that  $H \subset Q_n$ . Suppose that  $R_0, \dots, R_n$  are subgroups of  $G$  with  $R_i \subset Q_{i-1} \cap Q_i$  for  $i > 0$ , and  $R_0 \subset Q_0$ . Let  $\tilde{X} = Q_n \times Q_{n-1} \times \dots \times Q_0$ , and let  $X = Q_n \times_{R_n} Q_{n-1} \times_{R_{n-1}} \dots \times_{R_2} Q_1 \times_{R_1} Q_0 / R_0$ .

Let  $\pi : \tilde{X} \rightarrow X$  denote the quotient morphism. If  $\tilde{x} = (q_n, \dots, q_0) \in \tilde{X}$ , write  $[q_n, \dots, q_0]$  for  $\pi(\tilde{x})$ . The following lemma is a straightforward computation.

**Lemma 3.9.** *The point  $[q_n, \dots, q_0] \in X$  is  $H$ -fixed  $\Leftrightarrow$  for  $i = 0, \dots, n$ , we have  $C(q_i^{-1} q_{i+1}^{-1} \dots q_n^{-1})(H) \subset R_i$ .*

Assume that  $[q_n, \dots, q_0]$  is  $H$ -fixed. Let  $V = \mathfrak{q}_n \oplus \dots \oplus \mathfrak{q}_0$ . Define an  $H$ -module structure on  $V$  by making  $h \in H$  act on the  $\mathfrak{q}_i$  summand by  $\text{Ad}(C(q_i^{-1} q_{i+1}^{-1} \dots q_n^{-1})(h))$ . Then  $\mathfrak{r}_n \oplus \dots \oplus \mathfrak{r}_0$  is an  $H$ -submodule of  $V$ . This gives an  $H$ -module structure on  $V/(\mathfrak{r}_n \oplus \dots \oplus \mathfrak{r}_0) \cong \mathfrak{q}_n/\mathfrak{r}_n \oplus \dots \oplus \mathfrak{q}_0/\mathfrak{r}_0$ .

As an immediate consequence of the definitions of the  $H$ -module structures, we have the following lemma.

**Lemma 3.10.** *Assume that  $[q_n, \dots, q_0] \in X$  is  $H$ -fixed. The embedding  $\psi : \mathfrak{r}_n \oplus \dots \oplus \mathfrak{r}_0 \rightarrow V$  which takes  $(\xi_n, \dots, \xi_0)$  to*

$$(\xi_n, -\text{Ad}(q_{n-1}^{-1})\xi_n + \xi_{n-1}, -\text{Ad}(q_{n-2}^{-1})\xi_{n-1} + \xi_{n-2}, \dots, -\text{Ad}(q_0^{-1})\xi_1 + \xi_0)$$

*is  $H$ -equivariant.*

Let  $V_1$  denote  $\psi(\mathfrak{r}_n \oplus \cdots \oplus \mathfrak{r}_0)$ ; if  $[q_n, \dots, q_0]$  is  $H$ -fixed, then  $V_1$  is an  $H$ -submodule of  $V$ . Denote by  $L(q_n, \dots, q_0)$  the map of  $\tilde{X}$  to itself which sends  $(a_n, \dots, a_0)$  to  $(q_n a_n, \dots, q_0 a_0)$ . Let

$$\rho = \pi_* \circ L(q_n, \dots, q_0)_* : V \rightarrow T_{[q_n, \dots, q_0]} X. \quad (3.11)$$

**Lemma 3.12.** *Let  $x = [q_n, \dots, q_0] \in X$ . Let  $\xi \in \mathfrak{q}$ , and let  $\xi^\#$  denote the induced vector field on  $X$ . Then  $\xi_x^\# = \rho(\text{Ad}(q_n^{-1})\xi, 0, \dots, 0)$ .*

*Proof.* Let  $\phi$  be a function on  $X$ . Then

$$\begin{aligned} \xi_x^\# \phi &= \frac{d}{dt} \phi(\exp(t\xi)q_n, \dots, q_0) \\ &= \frac{d}{dt} \phi([q_n \exp(t\text{Ad}(q_n^{-1})\xi), \dots, q_0]). \end{aligned}$$

Tracing through the definitions shows that this equals

$$\rho(\text{Ad}(q_n^{-1})\xi, 0, \dots, 0)\phi.$$

□

The main purpose of this subsection is to prove the following proposition.

**Proposition 3.13.** *With notation as above, assume that  $x = [q_n, \dots, q_0]$  is an  $H$ -fixed point of  $X$ . The map  $\rho : V \rightarrow T_{[q_n, \dots, q_0]} X$  is  $H$ -equivariant with kernel  $V_1$ , and hence induces an  $H$ -module isomorphism  $V/V_1 \rightarrow T_{[q_n, \dots, q_0]} X$ .*

*Proof.* The proof is by induction on  $n$ . The case  $n = 0$  is handled by Proposition 3.6, taking  $M$  to be a point. Suppose that the proposition is true for  $n - 1$ . Let  $M = Q_{n-1} \times_{R_{n-1}} \cdots \times_{R_2} Q_1 \times_{R_1} Q_0/R_0$ . and let  $m = [q_{n-1}, \dots, q_0] \in M$ . Then we can identify  $X$  with  $Q_n \times_{R_n} M$  and the point  $x$  with  $[q_n, m]$ . Let

$$\rho_n : \mathfrak{q}_n \oplus T_m M \rightarrow T_x X$$

be the map defined in (3.3) (with  $Q_n$  in place of  $Q$ , and  $R_n$  in place of  $R$ ), and let

$$\rho_{n-1} : \mathfrak{q}_{n-1} \oplus \cdots \oplus \mathfrak{q}_0 \rightarrow T_m M$$

be the analog of the map  $\rho$  (with  $M$  in place of  $X$  and  $m$  in place of  $x$ ). We have

$$\mathfrak{q}_n \oplus \mathfrak{q}_{n-1} \oplus \cdots \oplus \mathfrak{q}_0 \xrightarrow{1 \times \rho_{n-1}} \mathfrak{q}_n \oplus T_m M \xrightarrow{\rho_n} T_x X,$$

and

$$\rho = \rho_n \circ (1 \times \rho_{n-1}).$$

By hypothesis,  $x = [q_n, m]$  is  $H$ -fixed. By Lemma 3.4, this implies that  $m$  is  $H' = C(q_n^{-1})(H)$ -fixed. By Proposition 3.6, the map  $\rho_n$  is  $H$ -equivariant, and our inductive hypothesis implies that  $\rho_{n-1}$  is  $H'$ -equivariant. Combining these, we can show that  $\rho$  is

$H$ -equivariant as follows. To simplify the notation, write  $r_i = C(q_i^{-1}q_{i+1}^{-1} \cdots q_n^{-1})(h) \in R_i$ . Then

$$\begin{aligned} L(h)_*\rho(\xi_n, \dots, \xi_0) &= L(h)_*\rho_n(\xi_n, \rho_{n-1}(\xi_{n-1}, \dots, \xi_0)) \\ &= \rho_n(\text{Ad}(r_n)\xi_n, L(r_n)_*\rho_{n-1}(\xi_{n-1}, \dots, \xi_0)) \end{aligned}$$

by  $H$ -equivariance of  $\rho_n$ . Using  $H'$ -equivariance of  $\rho_{n-1}$ , this equals

$$\rho_n(\text{Ad}(r_n)\xi_n, \rho_{n-1}(\text{Ad}(r_{n-1})\xi_{n-1}, \dots, \text{Ad}(r_0)\xi_0))$$

which (as  $\rho = \rho_n \circ (1 \times \rho_{n-1})$ ) is

$$\rho(\text{Ad}(r_n)\xi_n, \text{Ad}(r_{n-1})\xi_{n-1}, \dots, \text{Ad}(r_0)\xi_0) = \rho(h \cdot (\xi_n, \dots, \xi_0)).$$

Hence  $\rho$  is  $H$ -equivariant, as desired.

It remains to check that  $\ker \rho = V_1$ . We will make use of the following observation: if  $f : V \rightarrow W$  is a linear map of vector spaces, and if  $A \subset V$ ,  $B \subset W$  are subspaces such that  $f(A) = B$ , then  $f^{-1}(B) = A + \ker f$ . As  $\rho = \rho_n \circ (1 \times \rho_{n-1})$ , we see that

$$\ker \rho = (1 \times \rho_{n-1})^{-1}(\ker \rho_n).$$

By Proposition 3.6,

$$\ker \rho_n = \{(\xi, -\xi_m^\#) \mid \xi \in \mathfrak{r}_n\}.$$

By Lemma 3.12,

$$\rho_{n-1}(\text{Ad}(q_{n-1}^{-1})\xi, 0, \dots, 0) = \xi_m^\#.$$

Therefore, if we let

$$A = \{(\xi, -\text{Ad}(q_{n-1}^{-1})\xi, 0, \dots, 0) \mid \xi \in \mathfrak{r}_n\} \subset \mathfrak{q}_n \oplus \cdots \oplus \mathfrak{q}_0,$$

we see that  $(1 \times \rho_{n-1})(A) = \ker \rho_n$ . Therefore, by the observation above,

$$(1 \times \rho_{n-1})^{-1}(\ker \rho_n) = A + \ker(1 \times \rho_{n-1}) = A + (0 \oplus \ker \rho_{n-1}).$$

By our inductive hypothesis,

$$\ker \rho_{n-1} = \{(\xi_{n-1}, -\text{Ad}(q_{n-2}^{-1})\xi_{n-1} + \xi_{n-2}, \dots, -\text{Ad}(q_0^{-1})\xi_1 + \xi_0) \mid \xi_i \in \mathfrak{r}_i\}.$$

Hence, from the definition of  $V_1$ , we see that

$$A + (0 \oplus \ker \rho_{n-1}) = V_1,$$

so  $\ker \rho = V_1$ , as desired.  $\square$

Recall that  $\mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \mathfrak{q}_0/\mathfrak{r}_0$  has an  $H$ -module structure defined after Lemma 3.9.

**Corollary 3.14.** *Assume that the hypotheses of the preceding proposition hold, and in addition assume that  $H$  is reductive. Then  $T_{[q_n, \dots, q_0]}X$  and  $\mathfrak{q}_n/\mathfrak{r}_n \oplus \cdots \mathfrak{q}_0/\mathfrak{r}_0$  are isomorphic  $H$ -modules.*

*Proof.* This is proved by an argument similar to the proof of Corollary 3.8.  $\square$

It is convenient to give an alternative formulation of the corollary in case  $H$  is a torus and each  $q_i^{-1} \dots q_n^{-1}$  is in the normalizer  $N_G(H)$  of  $H$ . If  $V$  is a representation of the torus  $H$ , let  $\Phi(V)$  denote the multiset of weights of  $H$  acting on  $V$  (this is the set of weights, where each weight is counted with multiplicity). We adopt the convention that if  $s$  occurs  $a$  times in the multiset  $A$ , and  $b$  times in the multiset  $B$ , then  $s$  occurs  $a + b$  times in the multiset  $A \cup B$ .

The group  $N_G(H)$  acts on weights by the rule that if  $\lambda$  is a weight of  $H$  (so  $e^\lambda : H \rightarrow \mathbf{C}^*$  is a homomorphism), and  $w \in N_G(H)$ , then  $w\lambda$  is the weight satisfying

$$e^{w\lambda}(h) = e^\lambda(w^{-1}hw).$$

With these definitions, we can reformulate the preceding corollary as follows.

**Corollary 3.15.** *Assume the hypotheses of Proposition 3.13 hold. Assume in addition that  $H$  is a torus and that each  $q_i^{-1} \dots q_n^{-1}$  is in the normalizer of  $H$ . Then*

$$\Phi(T_{[q_n, \dots, q_0]}X) = q_n \cdot \Phi(\mathfrak{q}_n/\mathfrak{r}_n) \cup q_n q_{n-1} \cdot \Phi(\mathfrak{q}_{n-1}/\mathfrak{r}_{n-1}) \cup \dots \cup q_n q_{n-1} \dots q_0 \cdot \Phi(\mathfrak{q}_0/\mathfrak{r}_0).$$

*Proof.* This follows from Corollary 3.14 and the definition of the  $H$ -action on  $\mathfrak{q}_n/\mathfrak{r}_n \oplus \dots \oplus \mathfrak{q}_0/\mathfrak{r}_0$  given after Lemma 3.9.  $\square$

*Remark 3.16.* The results of this section can be generalized to remove the assumption that the  $Q_i$  are subgroups of a group  $G$ . Instead of  $R_i \subset Q_{i-1} \cap Q_i$ , what we need are inclusions  $R_i \subset Q_i$  and homomorphisms  $R_i \rightarrow Q_{i-1}$ .

**3.4. Induced vector bundles.** Keep the notation of the previous subsection. If  $V$  is a representation of  $R_0$ , then

$$\mathbf{V} = Q_n \times_{R_n} Q_{n-1} \times_{R_{n-1}} \dots \times_{R_1} Q_0 \times_{R_0} V \rightarrow X$$

is a  $Q_n$ -equivariant vector bundle. If  $x = [q_n, \dots, q_0]$  is fixed by  $H$ , then the fiber  $\mathbf{V}_x$  is a representation of  $H$ , which is described as follows.

**Proposition 3.17.** *With notation as above, if  $x = [q_n, \dots, q_0]$  is fixed by  $H$ , then as an  $H$ -module,  $\mathbf{V}_x$  is isomorphic to  $V$ , with  $H$ -action given by*

$$h \cdot v = C(q_0^{-1} \dots q_n^{-1})(h)v,$$

for  $h \in H$ ,  $v \in V$ . (Note that the right side of this equation makes sense, since  $C(q_0^{-1} \dots q_n^{-1})(h) \in R_0$  and  $V$  is a representation of  $R_0$ .)

*Proof.* We have an isomorphism  $f : V \rightarrow \mathbf{V}_x$  given by  $v \mapsto [q_n, \dots, q_0, v]$ . An element  $h \in H$  acts on  $\mathbf{V}_x$  by

$$\begin{aligned} h \cdot f(v) &= [hq_n, q_{n-1}, \dots, q_0, v] \\ &= [q_n C(q_n^{-1})(h), q_{n-1}, \dots, q_0, v] \\ &= [q_n, C(q_n^{-1})(h)q_{n-1}, \dots, q_0, v] \\ &\quad \vdots \\ &= [q_n, \dots, q_0, C(q_0^{-1} \cdots q_n^{-1})(h)v] \\ &= f(C(q_0^{-1} \cdots q_n^{-1})(h)v). \end{aligned}$$

The proposition follows.  $\square$

**Corollary 3.18.** *Keep the assumptions of the previous proposition. Assume in addition that  $H$  is a torus and that each  $q_i^{-1} \cdots q_n^{-1}$  is in the normalizer of  $H$ , so in particular  $H \subset R_0$ . Suppose that  $V$  is a 1-dimensional representation of  $R_0$  and that the weight of  $H$  on  $V$  (induced by the inclusion  $H \subset R_0$ ) is  $\lambda$ . Then the weight of  $H$  acting on  $\mathbf{V}_x$  is  $q_n \cdots q_0 \cdot \lambda$ .*

*Proof.* This follows immediately from the proposition and the definition of the action of the normalizer of  $H$  on weights of  $H$  given in the preceding subsection.  $\square$

#### 4. LOCALIZATION AND $\gamma_{\mathcal{Q}}^{-1}(f)$

We return to the situation of Section 2. Using the computations of Section 3, we apply localization theorems in equivariant cohomology and K-theory to  $\gamma_{\mathcal{Q}}^{-1}(f)$ . The first application gives a character formula for cohomology of  $\mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)$ . This is related to associated cycles of discrete series representations. The second application is to express the homology and K-theory classes determined by  $\gamma_{\mathcal{Q}}^{-1}(f) \subset \mathfrak{B}$  in terms of Schubert bases. This answers (for the components  $\gamma_{\mathcal{Q}}^{-1}(f)$ ) a question of Springer.

The first step is to determine the fixed points of the action of  $H$ . Recall that the fixed point set of  $H$  on  $\mathfrak{B}$  is  $W \cdot \mathfrak{b}$ , where  $W$  is the Weyl group of  $G$ . Since  $\gamma_{\mathcal{Q}}^{-1}(f)$  is a subset of  $\mathfrak{B}$ , the fixed points on  $\gamma_{\mathcal{Q}}^{-1}(f)$  are a subset of  $W \cdot \mathfrak{b}$ . In fact,  $\gamma_{\mathcal{Q}}^{-1}(f)$  is contained in the flag variety for  $K$ , so the fixed points are a subset of  $W(K) \cdot \mathfrak{b}$ .

**Proposition 4.1.** *The fixed point set of the action of  $H$  on  $\gamma_{\mathcal{Q}}^{-1}(f)$  is*

$$\{w_m \cdots w_1 w_0 \cdot \mathfrak{b} : w_i \in W(L_i)\}. \quad (4.2)$$

*Proof.* We prove the following statement by induction on  $k$ : the fixed point set of  $H$  on  $Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$  is

$$\{w_k \cdots w_1 w_0 \cdot \mathfrak{b} : w_i \in W(L_i)\}, \quad (4.3)$$

for  $k = 0, 1, \dots, m$ . For  $k = 0$  this is clear since  $Q_0 \cdot \mathfrak{b} = L_0 \cdot \mathfrak{b}$  is the flag variety for  $L_0$ . Assume that (4.3) holds for  $k - 1$ , and let  $w \cdot \mathfrak{b} = q_k \cdots q_1 q_0 \cdot \mathfrak{b}$  be a fixed point in

$Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$ . Using the fact (proved in Section 2) that  $F_k : X_k \rightarrow Q_k \cdots Q_1 Q_0 \cdot \mathfrak{b}$  is an  $H$ -equivariant bijection, we have

$$h[q_k, \dots, q_1, q_0] = [q_k, \dots, q_1, q_0] \in X_k, \text{ for all } h \in H.$$

Therefore,  $hq_k \equiv q_k$  modulo  $R_k$ , for all  $h \in H$ . In other words,  $q_k \cdot \mathfrak{r}_k$  is a fixed point of the action of  $H$  on the (generalized) flag variety  $Q_k/R_k = L_k/R_k \cap L_k$ . We conclude that  $q_k = w_k r_k$ , for some representative  $w_k$  of  $W(L_k)$  and  $r_k \in R_k$ . Therefore,

$$w_k^{-1} w \cdot \mathfrak{b} = r_k q_{k-1} \cdots q_1 q_0 \cdot \mathfrak{b}$$

is  $H$ -fixed. By the inductive hypothesis the righthand side is  $w_{k-1} \cdots w_1 w_0 \cdot \mathfrak{b}$ ,  $w_i \in W(L_i)$ . The proposition now follows.  $\square$

Recall that if  $V$  is an  $H$ -module then  $\Phi(V)$  denotes the set of weights of  $V$ , counted with multiplicity.

**Corollary 4.4.** *The weights of  $H$  on the tangent space to the  $H$ -fixed point  $w_m \cdots w_1 w_0 \cdot \mathfrak{b}$  of  $\gamma_Q^{-1}(f)$  (where  $w_i \in W(L_i)$ ) are*

$$w_n \cdot \Phi(\mathfrak{q}_n/\mathfrak{r}_n) \cup w_n w_{n-1} \cdot \Phi(\mathfrak{q}_{n-1}/\mathfrak{r}_{n-1}) \cup \cdots \cup w_n w_{n-1} \cdots w_0 \cdot \Phi(\mathfrak{q}_0/\mathfrak{r}_0).$$

*All the weights have multiplicity 1.*

*Proof.* The description of the weights is an immediate consequence of Corollary 3.15. Each weight has multiplicity 1 because if  $w \in W$ , then  $T_{w\mathfrak{b}}\gamma_Q^{-1}(f) \subset T_{w\mathfrak{b}}\mathfrak{B}$ , and  $H$  acts with multiplicity 1 on  $T_{w\mathfrak{b}}\mathfrak{B}$ .  $\square$

Recall that the representation ring  $R(H)$  is the free abelian group spanned by  $e^\mu$ , as  $\mu$  runs over all weights of  $H$ . The multiplication in  $R(H)$  is defined by the rule  $e^\lambda e^\mu = e^{\lambda+\mu}$ . Let  $\mu_1, \dots, \mu_n$  be weights of  $H$ . Let  $\mathbf{C}_{\mu_i}$  denote the 1-dimensional representation of  $H$  corresponding to  $\mu_i$ , and let  $V = \bigoplus_i \mathbf{C}_{\mu_i}$ . Define  $\lambda_{-1}(V) = \prod_i (1 - e^{\mu_i}) \in R(H)$ . We will make use of the following general fact, which is a consequence of the localization theorem in equivariant  $K$ -theory (see [7, Remark 5.11.8]).

**Proposition 4.5.** *Let  $H$  be a torus acting on a smooth complete algebraic variety  $M$ , and assume that the set  $M^H$  of  $H$ -fixed points on  $M$  is finite. Let  $L$  be an  $H$ -equivariant line bundle on  $M$ , and let  $\mathcal{L}$  denote the corresponding invertible sheaf on  $M$ . For  $m \in M^H$ , let  $\mu(m)$  denote the weight of  $H$  on  $L_m$ . Then in  $R(H)$ , we have*

$$\sum_i (-1)^i H^i(M, \mathcal{L}) = \sum_{m \in M^H} \frac{e^{\mu(m)}}{\lambda_{-1}(T_m^* M)}.$$

In this proposition, the individual terms in the sum on the right hand side are in the quotient field of  $R(H)$ ; they need not be in  $R(H)$ , but their sum is. Combining this proposition with our description of the weights of  $H$  on tangent spaces (Corollary 4.4), and Corollary 3.18, we obtain the following character formula (with notation as in Section 2).

**Theorem 4.6.** *Let  $\tau \in \mathfrak{h}^*$  be an integral weight. Let  $A_i$  denote the set of weights of  $H$  on  $\mathfrak{q}_i/\mathfrak{r}_i$ . In  $R(H)$ , we have*

$$\sum_i (-1)^i H^i(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau)) = \sum \frac{e^{w_m w_{m-1} \cdots w_0 \tau}}{\prod_i \prod_{\mu \in A_i} (1 - e^{-w_m w_{m-1} \cdots w_i \mu})}.$$

Here the sum is taken over all products  $w_m w_{m-1} \cdots w_0$ , where  $w_i \in W(L_i)$ .

Note that in this theorem the sum is not over  $m$ -tuples  $(w_m, \dots, w_0)$  where  $w_i \in W(L_i)$ , but rather over the distinct products  $w_m w_{m-1} \cdots w_0$ . Also, if  $\tau$  is dominant with respect to the positive system  $\Delta_{1,c}^+$ , then by the cohomology vanishing theorem (Theorem 2.34), Theorem 4.6 gives a formula for the character of the  $H$ -representation on  $H^0(\gamma_{\mathcal{Q}}^{-1}(f), \mathcal{O}_{\gamma_{\mathcal{Q}}^{-1}(f)}(\tau))$ .

Now we turn to the question of expressing homology classes of  $\gamma_{\mathcal{Q}}^{-1}(f)$  in terms of Schubert bases. Because  $\gamma_{\mathcal{Q}}^{-1}(f)$  is an  $H$ -invariant subvariety of  $\mathfrak{B}$ , it defines classes in the homology (or cohomology) and  $K$ -theory (ordinary or  $H$ -equivariant) of  $\mathfrak{B}$ . The homology and  $K$ -theory of  $\mathfrak{B}$  have Schubert bases, that is, bases defined in terms of Schubert varieties. Using localization theorems in equivariant cohomology and  $K$ -theory, known results about Schubert classes, and Corollary 4.4, we can express the equivariant classes determined by  $\gamma_{\mathcal{Q}}^{-1}(f)$  in terms of the Schubert bases. The expressions in ordinary homology or  $K$ -theory are obtained by specializing the corresponding  $H$ -equivariant expressions.

We begin by recalling some known facts about equivariant (co)homology and Schubert classes. The facts we need can be found in [15] or [22]. Given any space  $Z$  with  $H$ -action, one can define the equivariant cohomology groups  $H_H^i(Z)$  and the equivariant Borel-Moore homology groups  $H_i^H(Z)$ . Then  $H_*^H(Z) = \oplus_i H_i^H(Z)$  is a module for the ring  $H_H^*(Z) = \oplus_i H_H^i(Z)$ . We have  $H_H^*(\text{point}) = S(\hat{H})$ , the symmetric algebra on the group of characters of  $H$ . Note that  $S(\hat{H})$  is the polynomial ring  $\mathbf{Z}[\lambda_1, \dots, \lambda_n]$ , where  $\lambda_1, \dots, \lambda_n$  is a basis for the free abelian group  $\hat{H}$ . By pulling back from the map from  $Z$  to a point, there is a map  $S(\hat{H}) \rightarrow H_H^*(Z)$ , and thus  $H_*^H(Z)$  is an  $S(\hat{H})$ -module. If  $Z$  is a complete variety, there is a pairing

$$(\ , \ ) : H_H^*(Z) \otimes_{S(\hat{H})} H_*^H(Z) \rightarrow S(\hat{H}).$$

Now consider  $Z = \mathfrak{B}$ . By definition, the Schubert class  $X_w$  is the closure of  $B \cdot w\mathfrak{b}$  in  $\mathfrak{B}$ . Each  $X_w$  defines a class  $[X_w]_H$  in the equivariant Borel-Moore homology  $H_{\ell(w)}^H(\mathfrak{B})$ . The space  $H_*^H(\mathfrak{B})$  is a free  $S(\hat{H})$ -module with basis given by  $[X_w]_H$ . Moreover, there is a basis  $\{x_w\}$  of  $H_H^*(\mathfrak{B})$  with the property that

$$(x_u, [X_v]_H) = \delta_{u,v},$$

for all  $u, v \in W$ .

Given a class  $\eta \in H_H^*(\mathfrak{B})$ , and  $u \in W$ , write  $\eta(u)$  for the pullback of  $\eta$  to the equivariant cohomology group  $H_H^*(u\mathfrak{b})$ . Because  $u\mathfrak{b}$  is a point, this group is identified

with  $S(\hat{H})$ . The statement of Theorem 4.8 will involve the polynomials  $x_w(u)$ . These polynomials are known. Indeed, [22, Lemma 11.1.9] contains an explicit formula for elements  $R(w, u) \in S(\hat{H})$ . In Sections 11.1-11.3 of that book, especially Prop. 11.3.10, the connection of these elements with equivariant cohomology is explained; the result, in our notation, is that,

$$x_w(u) = (-1)^{\ell(w)} R(w, u)$$

(the sign  $(-1)^{\ell(w)}$  is necessary because we have taken the roots in  $\mathfrak{h}$  to be negative, the opposite of Kumar's convention).

Because  $\gamma_Q^{-1}(f)$  is  $H$ -invariant, it defines a class  $[\gamma_Q^{-1}(f)]_H \in H_{2d}^H(\mathfrak{B})$ , where  $d$  is the complex dimension of  $\gamma_Q^{-1}(f)$ . Thus, we can write

$$[\gamma_Q^{-1}(f)]_H = \sum_w A_w [X_w]_H.$$

Observe that each  $A_w$  is a polynomial in  $\lambda_1, \dots, \lambda_n$ . Let  $a_w$  be obtained from  $A_w$  by setting all the  $\lambda_i$  equal to 0. Then in the ordinary homology  $H_*(X)$  we have the equation

$$[\gamma_Q^{-1}(f)] = \sum_w a_w [X_w];$$

Springer's original question was essentially to calculate the coefficients  $a_w$ . The  $A_w$  can be calculated by pairing with the dual basis:

$$A_w = (x_w, [\gamma_Q^{-1}(f)]_H).$$

See [15] or [22] for proofs and references for the preceding facts.

If  $H$  acts on a smooth variety  $M$  and  $u$  is an  $H$ -fixed point, let  $P_u(M) \in S(\hat{H})$  denote the product of the weights (with multiplicity) of  $H$  on  $T_u M$ . (As in Corollary 4.4, if  $M$  is an  $H$ -invariant subvariety of  $\mathfrak{B}$ , then all weights of  $T_u M$  must occur with multiplicity 1.) The following proposition gives a formula for the pairing.

**Proposition 4.7.** *Let  $M$  be a smooth closed  $H$ -invariant subvariety of  $\mathfrak{B}$ , and let  $\eta \in H_H^*(\mathfrak{B})$ . Then*

$$(\eta, [M]_H) = \sum_{u \in W} \frac{\eta(u)}{P_u(M)},$$

where we have written  $P_u(M)$  for  $P_{ub}(M)$ . (The individual terms of the sum on the right hand side are in the quotient field  $Q(\hat{H})$  of  $S(\hat{H})$ , but the sum is in  $S(\hat{H})$ ).

*Proof.* This is an immediate consequence of the “integration formula” of [8, Cor. 1] (the formula given there is an algebraic version of [1, Equation (3.8)]). In that paper, the calculation is done in equivariant Chow groups, but the same formula holds in equivariant Borel-Moore homology (in fact, for  $\mathfrak{B}$  the two theories coincide because  $\mathfrak{B}$  is paved by affines).  $\square$

As an immediate consequence, we obtain the following theorem.

**Theorem 4.8.** *The class  $[\gamma_Q^{-1}(f)]_H$  in  $H_*^H(\mathfrak{B})$  is given by  $\sum_w A_w [X_w]_H$ , where*

$$A_w = \sum_u \frac{x_w(u)}{P_u(\gamma_Q^{-1}(f))}.$$

Here the sum is over all  $u \in W$  which can be written as  $u = w_m w_{m-1} \cdots w_0$ , where  $w_i \in W(L_i)$ . Also, if  $A_i$  denotes the set of weights of  $H$  on  $\mathfrak{q}_i/\mathfrak{r}_i$ , then fixing an expression of  $u$  as a product  $w_m w_{m-1} \cdots w_0$ , we have

$$P_u(\gamma_Q^{-1}(f)) = \prod_i \prod_{\mu \in A_i} w_m w_{m-1} \cdots w_i \mu.$$

We can perform the analogous calculations in equivariant K-theory using almost identical arguments. We begin by recalling a some facts about the equivariant K-theory of the flag variety; see [16] for more details and references. If  $Z$  is a scheme with an  $H$ -action,  $K_H(Z)$  denotes the Grothendieck group of  $H$ -equivariant coherent sheaves on  $Z$ . This is a module for the representation ring  $R(H)$ , which we recall is the free abelian group spanned by  $e^\mu$ , for  $\mu \in \hat{H}$ . If  $Z$  is smooth, then every equivariant coherent sheaf on  $Z$  admits a finite equivariant resolution by locally free sheaves, so  $K_H(Z)$  can be identified with the Grothendieck group of  $H$ -equivariant vector bundles on  $Z$ . In this case,  $K_H(Z)$  has a ring structure induced from the tensor product of vector bundles. If  $Z$  is complete, there is a pairing

$$(\ , \ ) : K_H(Z) \otimes_{R(H)} K_H(Z) \rightarrow R(Z)$$

defined by

$$(v_1, v_2) = \sum_i (-1)^i H^i(Z, v_1 \cdot v_2).$$

The group  $K_H(\mathfrak{B})$  is a free  $R(H)$ -module with basis  $\{[\mathcal{O}_{X_w}]\}_{w \in W}$ , that is, a basis given by the classes of structure sheaves of Schubert varieties. Just as in cohomology, there is a dual basis  $\{\xi^w\}_{w \in W}$ , characterized by the property that

$$(\xi^u, [\mathcal{O}_{X_w}]) = \delta_{u,w}.$$

Write  $\xi^u(w)$  for the pullback of  $\xi^u$  to the equivariant K-theory group  $K_H(w\mathfrak{b}) \cong R(H)$ . Observe that the  $\xi^u(w)$  are known; [14] and [32] give explicit formulas for these classes (Willems uses a slightly different basis, but see [16] for a description of the relations between various bases).

If  $w\mathfrak{b}$  is an  $H$ -fixed point of  $\gamma_Q^{-1}(f)$ , we can write  $w = w_m w_{m-1} \cdots w_0$ , where  $w_i \in W(L_i)$ . Define

$$Q_w(\gamma_Q^{-1}(f)) = \lambda_{-1}(T_{w\mathfrak{b}}^* \gamma_Q^{-1}(f)) = \prod_i \prod_{\mu \in A_i} (1 - e^{-w_m w_{m-1} \cdots w_i \mu}),$$

where  $A_i$  is as in Theorem 4.8.

The expansion of the class  $[\mathcal{O}_{\gamma_Q^{-1}(f)}]$  in terms of the classes  $[\mathcal{O}_{X_w}]$  is given by the following theorem.

**Theorem 4.9.** *The class  $[\mathcal{O}_{\gamma_Q^{-1}(f)}]$  in  $K_H(\mathfrak{B})$  is given by  $\sum_u B_u[\mathcal{O}_{X_u}]$ , where*

$$B_u = \sum_w \frac{\xi^u(u)}{Q_w(\gamma_Q^{-1}(f))}.$$

*The sum is over all  $w \in W$  that can be written as  $w = w_m w_{m-1} \cdots w_0$ , where  $w_i \in W(L_i)$ . The corresponding expansion in the ordinary  $K$ -theory  $K(\mathfrak{B})$  is given by setting all the  $e^\lambda$  in each  $B_u$  equal to 1.*

*Proof.* The proof of this theorem is almost the same as the proof of Theorem 4.8, with  $K$ -theory in place of cohomology. We omit the details.  $\square$

*Remark 4.10.* There is a similar expansion for  $[\mathcal{O}_{\gamma_Q^{-1}(f)}]$  in terms of  $\xi^w$ , obtained by replacing  $\xi^u(u)$  by  $[\mathcal{O}_{X_w}](u)$  in the above formula. An explicit expression for the elements  $[\mathcal{O}_{X_w}](u)$  is given in [14] (see [21] for another proof).

## APPENDIX A.

### THE STANDARD TABLEAUX FOR COMPONENTS ASSOCIATED TO CLOSED ORBITS

A parametrization of the components of a Springer fiber for  $GL(n)$  is given in [29]. This is done in a natural way by associating to each component of  $\mu^{-1}(f)$  a standard tableau. Since this parametrization has become somewhat standard, in this appendix we identify the standard tableaux of the components studied in this article (and in [3]).

We mention that the results of this appendix may be found in the literature by piecing together a number of results about irreducible Harish-Chandra modules and their annihilators. For  $SU(p, q)$  the set of irreducible Harish-Chandra-modules of a given regular integral infinitesimal character is in one-to-one correspondence with  $K$ -orbits in  $\mathfrak{B}$ . On the other hand the primitive ideals in the enveloping algebra are parametrized in terms of standard tableau. In [13] an algorithm is given to associate to the  $K$ -orbit  $\mathcal{Q}$  corresponding to a Harish-Chandra module  $X$  the standard tableau corresponding to  $\text{Ann}(X)$ . The algorithm in fact also constructs a signed tableau having the same shape as the standard tableau. As shown in [30], the signed tableau is the associated variety of  $X$  and the standard tableau is the component of the Springer fiber of a generic element for  $T_{\mathcal{Q}}^* \mathfrak{B}$ . Therefore, the statements of this appendix follow from [13] and [30]. We include the appendix as an alternative that is elementary and purely in terms of the geometric description of  $\gamma_Q^{-1}(f)$ .

We begin by recalling the parametrization in [29]. Suppose that  $f$  is any nilpotent element of  $\mathfrak{g} = \mathfrak{gl}(n)$ . The Young diagram associated to  $f$  consists of rows of boxes; the lengths of these rows are the sizes of the Jordan blocks of  $f$  (which are equal to the lengths of the strings, in the language of Section 1). By a standard tableau we mean a numbering of a Young diagram (having  $n$  boxes) by the numbers in  $\{1, 2, \dots, n\}$  in such a way that the numbers increase from left to right along any row and down any column. The irreducible components of  $\mu^{-1}(f)$  are parametrized by the standard

tableaux having the same shape as the Young diagram of  $f$ . We will need to describe this parametrization carefully.

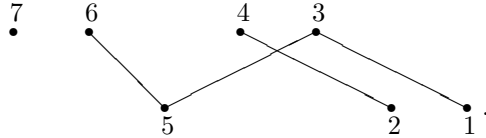
Suppose  $\mathfrak{b} \in \mu^{-1}(f)$  (i.e.,  $\mathfrak{b}$  is a Borel subalgebra containing  $f$ ). Let  $(E_i)$  be the flag  $\{0\} \subset E_1 \subset \cdots \subset E_n = \mathbf{C}^n$  that corresponds to  $\mathfrak{b}$  (that is,  $\mathfrak{b}$  is the stabilizer of  $(E_i)$ ). Then we may associate to  $(\mathfrak{b}, f)$  a standard tableau  $ST(\mathfrak{b}, f)$  as follows. The tableau of  $f|_{E_i}$  is obtained from the tableau of  $f|_{E_{i-1}}$  by attaching a new box onto the end of a row of the tableau of  $f|_{E_{i-1}}$ , or perhaps by starting a new row with one box, and putting the number  $i$  in the new box. The standard tableau of  $f|_{E_i}$  is obtained from that of  $f|_{E_{i-1}}$  by inserting the number  $i$  in the new block. The starting point for this procedure is a single block containing the number 1, corresponding to  $f|_{E_1}$ . The statement of [29] is that for each standard tableau  $ST$  having the same shape as the Young diagram of  $f$  there exists a unique irreducible component  $C$  of  $\mu^{-1}(f)$  so that for  $\mathfrak{b}$  in a dense open subset of  $C$ ,  $ST(\mathfrak{b}, f) = ST$ . This gives a one-to-one correspondence between irreducible components and standard tableaux.

An example is to take  $f$  as in the example of Section 1. Let  $\mathfrak{b}$  be as in that example. Then  $\mathfrak{b}$  is the stabilizer of the flag  $(E_i)$  with  $E_i$  the span of the standard basis vectors  $e_j$  with  $j$  among the labels of the  $i$  dots farthest to the right in the array. Then the associated standard tableau is

1	3	5	6
2	4		
7			

For the generic elements  $f$  considered in this paper, there is a simple algorithm for computing  $ST(\mathfrak{b}, f)$  in terms of the array. The algorithm is as follows. Relabel the dots in the array with  $1, 2, \dots, n$  in *decreasing* order from left to right. Each row in the tableau of  $f$  corresponds to a string or a dot not passed through by a string. Now fill in the boxes of each row with the new labels of the dots in the corresponding string. The rows with one box get the new labels of the dots in the array that do not lie in a string (in increasing order).

In the previous example, with all strings sketched in, we get



One now easily reads off the standard tableau obtained earlier.

The next proposition implies that in the preceding example, the component  $\gamma_Q^{-1}(f)$  corresponds to the standard tableau given above. To prove this one would need to find

an open dense set of Borel subalgebras  $\mathfrak{b}'$  in  $\gamma_Q^{-1}(f)$  for which  $ST(\mathfrak{b}', f)$  is this same standard tableau. This is how the proof of the proposition proceeds.

**Proposition A.1.** *Given a closed  $K$ -orbit  $\mathfrak{B}$  with base point  $\mathfrak{b}$  as in Section 1.1, let  $f \in \mathfrak{n}^- \cap \mathfrak{p}$  be the generic element constructed in Section 1. Then the standard tableau associated to  $\gamma_Q^{-1}(f)$  is  $ST(\mathfrak{b}, f)$ .*

To prove the proposition it suffices to prove Proposition A.5 below. We begin with two lemmas.

**Lemma A.2.** *Suppose  $\mathfrak{b}, \mathfrak{b}' \in \mu^{-1}(f)$  and  $\mathfrak{b}$  and  $\mathfrak{b}'$  correspond to flags  $(E_i)$  and  $(E'_i)$ . Then  $ST(\mathfrak{b}, f) = ST(\mathfrak{b}', f)$  if and only if  $\dim(\ker(f^k|_{E_i})) = \dim(\ker(f^k|_{E'_i}))$  for each  $i = 1, 2, \dots, n$  and each  $k \in \mathbf{Z}_{>0}$ .*

*Proof.* It follows from the above discussion that  $ST(\mathfrak{b}, f)$  is determined by the Jordan forms of  $f|_{E_i}$ ,  $i = 1, 2, \dots, n$ . Therefore it is determined by  $\dim(\ker(f^k|_{E_i}))$  for each  $i = 1, 2, \dots, n$  and each  $k \in \mathbf{Z}_{>0}$ .  $\square$

Define  $W_i = \text{span}_{\mathbf{C}}\{e_j : j \text{ is a label of a dot in the } i^{\text{th}} \text{ string}\}$ , for  $i = 1, \dots, m$ . List the standard basis vectors not in any  $W_i$  as  $e_{j_1}, \dots, e_{j_{r-m}}$  (some  $r$ ) with  $j_1 \geq j_2 \geq \dots$ . Now set  $W_{m+i} = \mathbf{C}e_{j_i}$ . It follows that  $\mathbf{C}^n = W_1 \oplus \dots \oplus W_r$  and each  $W_i$  is  $f$ -stable.

Consider a closed  $K$ -orbit  $Q = K \cdot \mathfrak{b}$  and the generic  $f$  in  $\mathfrak{n}^- \cap \mathfrak{p}$  as in Section 1.1. Let  $\mathfrak{v}_1^-, \dots, \mathfrak{v}_m^-$  be the nilradicals of the parabolic subalgebras  $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$  of  $\mathfrak{l}_i$ , as discussed in Lemma 1.9. Let  $\mathfrak{v}_0^- = \mathfrak{l}_0 \cap \mathfrak{n}^-$ . Let  $\mathfrak{v}_0, \mathfrak{v}_1, \dots, \mathfrak{v}_m$  be the nilradicals of the opposite parabolic subalgebras. Then  $\mathcal{V} := \exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1) \exp(\mathfrak{v}_0) \cdot \mathfrak{b}$  is dense and open in  $\gamma_Q^{-1}(f)$ .

**Lemma A.3.** *Suppose  $X_l \in \mathfrak{v}_l$  and  $j$  is arbitrary. Then  $\exp(X_l)e_j = e_j + y$ , where  $y$  is a linear combination of standard basis vectors  $e_a$  with  $a < j$  and  $a$  in the same  $\mathfrak{g}_l$ -block as  $j$ .*

*Proof.* Consider  $X_l \in \mathfrak{v}_l$ . Then  $X_l$  is a linear combination of root vectors for  $\epsilon_a - \epsilon_b$  with  $a, b$  in the same  $\mathfrak{g}_l$ -block and  $a < b$ . Then  $X_l(e_j)$  is 0 if  $j$  is not in a  $\mathfrak{g}_l$ -block and is a linear combination of  $e_a$  with  $a$  in the same  $\mathfrak{g}_l$ -block as  $j$  and  $a < j$ . The lemma follows.  $\square$

**Lemma A.4.** *For any  $v \in \exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1) \exp(\mathfrak{v}_0)$ ,  $\ker(f^k) = \ker(f^k \circ v)$ .*

*Proof.* Since  $v$  is invertible,  $\ker(f^k)$  and  $\ker(f^k \circ v)$  have the same dimension. So it is enough to show that  $\ker(f^k) \subset \ker(f^k \circ v)$ . The following observation will be used. If  $a, b$  are in the same  $\mathfrak{g}_l$ -block (for some  $l$ ) and  $a < b$ , then  $f^k(e_b) = 0$  implies  $f^k(e_a) = 0$ . To see this it suffices to assume that  $l = 0$ . Then  $a$  lies in a later string than the string of  $b$  (since strings pass through the rightmost dot in a block). The statement follows from the construction of  $f$ .

$$\exp(X_l)\ker(f^k) \subset \ker(f^k).$$
$$\exp(X_l) \cdots \exp(X_0) \ker(f^k) \subset \ker(f^k), \text{ all } l.$$
$$v(\ker(f^k)) \subset \ker(f^k).$$

□

$$\ker(f^k|_{E'_i}) = v(\ker((f^k \circ v)|_{E_i})) = v(\ker(f^k|_{E_i})),$$

9

Consider the Young diagram

$$T = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

Let  $\mathcal{O}_T$  be the corresponding nilpotent orbit in  $\mathfrak{g}$ . We write down all  $\gamma_{\mathcal{Q}}^{-1}(f)$  that occur as components in a Springer fiber for an element of  $\mathcal{O}_T$ . There are several pairs  $(G, K)$  to consider.

When  $(p, q) = (1, 7)$  or  $(7, 1)$ , any array has just one dot in one of the two rows. Therefore no string has length 4, and a generic  $f$  cannot lie in  $\mathcal{O}_T$ . A similar argument shows the same holds when  $(p, q) = (2, 6)$  or  $(6, 2)$ .

Now consider  $(p, q) = (3, 5)$ . There are four closed orbits  $\mathcal{Q}$  with generic elements  $f \in \mathcal{O}_T$ . We list below these orbits  $\mathcal{Q}$  (by giving the corresponding  $\lambda \in \mathfrak{h}^*$ ), the standard tableau for  $\gamma_{\mathcal{Q}}^{-1}(f)$ , and the signed tableau corresponding to  $K \cdot f$ . (The orbits of  $K$  on  $\mathcal{N}_{\theta}$  are parametrized by signed tableaux. See [3] for a discussion of these signed tableaux, and how the tableau corresponding to a generic  $f$  is obtained from the array.)

$\lambda = (8, 4, 3 7, 6, 5, 2, 1)$	<table><tr><td>1</td><td>3</td><td>5</td><td>8</td></tr><tr><td>2</td><td>4</td><td>6</td><td></td></tr><tr><td>8</td><td></td><td></td><td></td></tr></table>	1	3	5	8	2	4	6		8				<table><tr><td>-</td><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td></td><td></td><td></td></tr></table>	-	+	-	+	-	+	-		-			
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Now consider  $(p, q) = (4, 4)$ ; there are again four cases.

$\lambda = (8, 5, 4, 3 7, 6, 2, 1)$	<table><tr><td>1</td><td>3</td><td>6</td><td>8</td></tr><tr><td>2</td><td>4</td><td>7</td><td></td></tr><tr><td>5</td><td></td><td></td><td></td></tr></table>	1	3	6	8	2	4	7		5				<table><tr><td>-</td><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td><td></td></tr><tr><td>+</td><td></td><td></td><td></td></tr></table>	-	+	-	+	-	+	-		+			
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$$\lambda = (6, 5, 4, 1|8, 7, 3, 2)$$

1	2	4	7
3	5	8	
6			

+	-	+	-
-	+	-	
+			

(Note that the last two come from the first two by interchanging the rows of the array.)

For  $(p, q) = (5, 3)$  there are four cases, each obtained from the  $(3, 5)$  case by interchanging the rows of the array. We get the following.

$$\lambda = (7, 6, 5, 2, 1|8, 4, 3)$$

1	3	5	8
2	4	6	
8			

+	-	+	-
+	-	+	
+			

$$\lambda = (7, 6, 3, 2, 1|8, 5, 4)$$

1	4	6	8
2	5	7	
3			

+	-	+	-
+	-	+	
+			

$$\lambda = (8, 7, 6, 3, 2|5, 4, 1)$$

1	2	4	6
3	5	7	
8			

-	+	-	+
+	-	+	
+			

$$\lambda = (8, 7, 4, 3, 2|6, 5, 1)$$

1	2	5	7
3	6	8	
4			

-	+	-	+
+	-	+	
+			

In summary, there are 12 closed orbits  $\mathcal{Q}$  with generic element in  $\mathcal{O}_T$ . The 12 orbits pair off (under the symmetry of interchanging the rows of the arrays) into pairs giving the same components, so there are six components of the type  $\gamma_{\mathcal{Q}}^{-1}(f)$ . A further observation is that for  $(p, q) = (3, 5)$  (or  $(5, 3)$ ), of the four orbits  $\mathcal{Q}$ , two orbits  $K \cdot f \subset \mathcal{N}_{\theta}$  occur and all four components are different. This is not the case for  $(p, q) = (4, 4)$ .

Similar statements may be made in general. Let us fix  $G = GL(n)$  and consider all pairs  $(G, K_{p,q})$  with  $K_{p,q} = GL(p) \times GL(q)$ ,  $p + q = n$ . Fix a Young diagram  $T$  and corresponding nilpotent orbit  $\mathcal{O}_T$  in  $\mathfrak{g}$ . The key observation is the following.

**Lemma A.6.** *Suppose  $\mathcal{Q} = K_{p,q} \cdot \mathbf{b}$  is a closed  $K_{p,q}$ -orbit and  $\mathcal{Q}' = K_{p',q'} \cdot \mathbf{b}'$  is a closed  $K_{p',q'}$ -orbit, and  $f$  and  $f'$  are the corresponding generic elements. Then  $ST(\mathbf{b}, f) = ST(\mathbf{b}', f')$  if and only if either  $\mathcal{Q} = \mathcal{Q}'$ , or  $(p, q) = (q', p')$  and the array for  $\mathcal{Q}$  is obtained from the array for  $\mathcal{Q}'$  by switching the two rows.*

*Proof.* This is clear from our description of  $ST(\mathbf{b}, f)$ , since there are two ways to reconstruct an array from a standard tableau (one with the last dot in the upper row and one with it in the lower row).  $\square$

The lemma implies the following proposition.

**Proposition A.7.** *Let  $(G, K)$  be one of our pairs  $(GL(n), GL(p)) \times GL(q)$ . For each Young diagram  $T$ , the components of the Springer fiber for  $\mathcal{O}_T$  of the form  $\gamma_Q^{-1}(f)$ , with  $Q$  a closed  $K$ -orbit and  $f$  generic, are distinct when  $p \neq q$ . When  $p = q$  there are an even number of such components, and each occurs for exactly two  $Q$ .*

## REFERENCES

- [1] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.
- [2] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
- [3] L. Barchini and R. Zierau, *Certain components of Springer fibers and associated cycles for discrete series representations of  $SU(p, q)$* , Representation Theory **12** (2008), 403–434, with appendix by P. Trapa.
- [4] A. Borel, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Math, vol. 126, Springer, New York, 1991.
- [5] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation Theories and Algebraic Geometry (Montreal, PQ, 1997) (Dordrecht), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., 1998, pp. 1–37.
- [6] J.-T. Chang, *Characteristic cycles of discrete series for  $\mathbf{R}$ -rank one groups*, Trans. A.M.S. **341** (1994), no. 2, 603–622.
- [7] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhauser Boston, Boston, MA, 1997.
- [8] D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619–636.
- [9] L. Fresse, *Betti numbers of Springer fibers in type A*, (2007), preprint (arXiv:0706.3656v1).
- [10] ———, *Singular components of the Springer fiber in the two-column case*, (2008), preprint (arXiv:0803.2188v1).
- [11] ———, *Fixed points of the torus on Springer fibers in the hook, two-rows and two-columns cases*, (2008), preprint (arXiv:0803.2183).
- [12] F. Fung, *On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory*, Adv. in Math. **178** (2003), 244–276.
- [13] D. Garfinkle, *The annihilators of irreducible Harish-Chandra modules for  $SU(p, q)$  and other type  $A_{n-1}$  groups*, Amer. J. Math. **115** (1993), no. 2, 305–369.
- [14] W. Graham, *Equivariant  $K$ -theory and Schubert varieties*, preprint.
- [15] ———, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.
- [16] W. Graham and S. Kumar, *On positivity in  $T$ -equivariant  $K$ -theory of flag varieties*, Int. Math. Res. Not. IMRN **2008** (2008).
- [17] A. Grothendieck, *Séminaire de géométrie algébrique. Revêtements étales et groupe fondamental*, Lecture Notes in Math., vol. 224, Springer-Verlag, Heidelberg, 1971.
- [18] J. J. Güemes, *On the homology classes for the components of some fibres of Springer’s resolution*, Orbites unipotentes et représentations, III, Astérisque, no. 173-174, 1989, pp. 257–269.
- [19] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., vol. 52, Springer Verlag, 1977.
- [20] B. Iverson, *A fixed point formula for action of tori on algebraic varieties*, Inventiones Math. **16** (1972), 229–136.
- [21] A. Knutson, *Schubert patches degenerate to subword complexes*, Transformation Groups **13** (2008), 715–726.
- [22] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser, Boston, 2002.
- [23] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), no. 2, 169–178.

- [24] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, 1994.
- [25] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, 1982.
- [26] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.
- [27] ———, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 279–293.
- [28] ———, *Contribution to Open problems in algebraic groups*, Katata (1983), Taniguchi Foundation.
- [29] R. Steinberg, *An occurrence of the Robinson-Schensted correspondence*, Journal of Algebra **113** (1988), 523–528.
- [30] P. Trapa, *Generalized Robinson-Schensted algorithms for real Lie groups*, International Math. Res. Not. (1999), no. 15, 803–834.
- [31] J. Vargas, *Fixed points under the action of unipotent elements of  $SL(n)$  in the flag variety*, Bol. Soc. Mat. Mexicana **24** (1979), no. 1, 1–14.
- [32] M. Willems, *Cohomologie et  $K$ -théorie équivariantes des variétés de Bott-Samelson et des variétés de drapeaux*, Bull. Soc. Math. France **132** (2004), no. 4, 569–589.

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